

A NEW $\frac{1}{2}$ -RICCI TYPE FORMULA ON THE SPINOR BUNDLE AND APPLICATIONS

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ABSTRACT. Consider a Riemannian spin manifold (M^n, g) ($n \geq 3$) endowed with a non-trivial 3-form $T \in \Lambda^3 T^*M$, such that $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{1}{2}T$ is the metric connection with skew-torsion T . In this note we introduce a generalized $\frac{1}{2}$ -Ricci type formula for the spinorial action of the Ricci endomorphism $\text{Ric}^s(X)$, induced by the one-parameter family of metric connections $\nabla^s := \nabla^g + 2sT$. This new identity extends a result described by Th. Friedrich and E. C. Kim, about the action of the Riemannian Ricci endomorphism on spinor fields, and allows us to present a series of applications. For example, we describe a new alternative proof of the generalized Schrödinger-Lichnerowicz formula related to the square of the Dirac operator D^s , induced by ∇^s , under the condition $\nabla^c T = 0$. In the same case, we provide integrability conditions for ∇^s -parallel spinors, ∇^c -parallel spinors and twistor spinors with torsion. We illustrate our conclusions for some non-integrable structures satisfying our assumptions, e.g. Sasakian manifolds, nearly Kähler manifolds and nearly parallel G_2 -manifolds, in dimensions 5, 6 and 7, respectively.

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1. INTRODUCTION

Let (M^n, g) ($n \geq 3$) be a connected Riemannian spin manifold endowed with a non-trivial 3-form $T \in \Lambda^3 T^*M$. Consider the one-parameter family of connections $\{\nabla^s : s \in \mathbb{R}\}$, given by

$$\nabla^s = \nabla^g + 2sT.$$

This is a line of metric connections with totally skew-symmetric torsion $T^s = 4sT$, which joins the connection $\nabla^{1/4} \equiv \nabla^c$ with torsion T , with the Levi-Civita connection $\nabla^0 \equiv \nabla^g$. By an abuse of notation next we shall refer to ∇^c by the term “characteristic connection”. Let us denote by Ric^s the Ricci tensor induced by ∇^s . In this note we focus on the action of the associated Ricci endomorphism $\text{Ric}^s(X)$ ($X \in \Gamma(TM)$), on the corresponding spinor bundle $\Sigma^g M$. Under the condition $\nabla^c T = 0$, and for any arbitrary spinor field $\varphi \in \mathcal{F}^g := \Gamma(\Sigma^g M)$, we show that this action can be described in terms of the Dirac operator D^s ($s \in \mathbb{R}$) induced by ∇^s . This takes place in Section 3, where we provide the following (*generalized*) $\frac{1}{2}$ -Ricci type formula (see Lemma 3.1)

$$(1.1) \quad \begin{aligned} \frac{1}{2} \text{Ric}^s(X) \cdot \varphi &= D^s(\nabla_X^s \varphi) - \nabla_X^s(D^s \varphi) - \sum_{j=1}^n e_j \cdot \left[\nabla_{\nabla_{e_j}^s X}^s \varphi + 4s \nabla_{T(X, e_j)}^s \varphi \right] \\ &\quad + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi, \end{aligned}$$

for any arbitrary vector field $X \in \Gamma(TM)$, spinor field $\varphi \in \mathcal{F}^g$ and $s \in \mathbb{R}$, where σ_T is the 4-form

$$\sigma_T := \frac{1}{2} \sum_{i=1}^n (e_i \lrcorner T) \wedge (e_i \lrcorner T).$$

From now on we shall mainly refer to (1.1) by the term $\frac{1}{2}$ -Ricci^s-formula, or $\frac{1}{2}$ -Ricci^s-identity. This can be viewed as the analogue of the *Riemannian* $\frac{1}{2}$ -Ricci formula, or in short $\frac{1}{2}$ -Ricci^g-formula, introduced by Friedrich and Kim in [17, Lem. 1.2]. The latter relates the Ricci endomorphism of the Levi-Civita connection with the Riemannian Dirac operator, i.e.

$$(1.2) \quad \frac{1}{2} \text{Ric}^g(X) \cdot \varphi = D^g(\nabla_X^g \varphi) - \nabla_X^g(D^g \varphi) - \sum_{j=1}^n e_j \cdot \nabla_{\nabla_{e_j}^g X}^g \varphi.$$

In [17] it was shown that the $\frac{1}{2}$ -Ric^g-identity is stronger than the Schrödinger-Lichnerowicz formula associated to the Riemann Dirac operator $D^g \equiv D^0$, in the sense that the first formula induces the second one, after a contraction. Here, we extend this result by proving that the new $\frac{1}{2}$ -Ric^s-formula induces the corresponding generalized formula of Schrödinger-Lichnerowicz type, associated to the Dirac operator D^s (see for example [14, Thm. 3.1], [4, Thm. 6.1] or [1, Thm. 3.2]), under the condition $\nabla^c T = 0$. Therefore, when the torsion form T is ∇^c -parallel we provide a new proof for this fundamental formula which is different than the traditional proofs, compare for instance with [14, 1, 4].

The new $\frac{1}{2}$ -Ricci type identity, being stronger than the generalized SL-formula for D^s , has several nice applications. In fact, it is a spinorial identity which reproduces all $\frac{1}{2}$ -Ricci type formulas associated to ∇^s (in the sense of [17]), even for $s = 0$, but also other known results. For example, in [9] we have recently introduced a *twistorial* $\frac{1}{2}$ -Ric^s-formula for *twistor spinors with torsion* with respect to the family ∇^s . Such spinors are elements in the kernel of the Penrose operator \mathcal{P}^s , induced by ∇^s . When T is ∇^c -parallel and M^n is compact, in [2, Corol. 3.2] it was shown that twistor spinors with torsion realize the equality case of an estimate for the first eigenvalue of the square of the cubic Dirac operator $D^{1/12} = D^g + \frac{1}{4}T$, under some additional geometric assumptions (e.g. constant scalar curvature). The twistorial $\frac{1}{2}$ -Ric^s-formula ([9, Lem. 2.2]) appears also under the condition $\nabla^c T = 0$ and in the context of spin geometry with (parallel) skew-torsion, it establishes the analogue of a basic result of Lichnerowicz [22] (see also [11, p. 123] or [19, Prop. A.2.1.(3a)]). In Section 3 we obtain the twistorial $\frac{1}{2}$ -Ric^s-formula via a new and easier method, in particular we prove that it coincides with the restriction of the $\frac{1}{2}$ -Ric^s-identity to the kernel of the twistor operator \mathcal{P}^s (see Theorem 3.5).

Next we proceed with an examination of ∇^c -parallel spinors and more general ∇^s -parallel spinors. Recall that when ∇^c is the characteristic connection of a non-integrable G -structure on (M^n, g) (in terms for example of [14]), then the condition $\nabla^c \varphi = 0$ for some non-trivial spinor field φ , imposes restrictions to the holonomy group $\text{Hol}(\nabla^c) \subset G$. Here, when T is ∇^c -parallel, we deduce that a non-trivial spinor field $\varphi_0 \in \mathcal{F}^g$ which is parallel with respect to ∇^s for some parameter $s \in \mathbb{R}$, must satisfy the following equations (for any $X \in \Gamma(TM)$ and for the same s)

$$\text{Ric}^s(X) \cdot \varphi_0 = 2s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi_0, \quad \text{Scal}^s \cdot \varphi_0 = -8s(3 - 4s)\sigma_T \cdot \varphi_0.$$

For $s = 0$ this yields the well-known Ric^g-flatness of (M^n, g) , while for ∇^c -parallel spinors we obtain the conditions given by Friedrich and Ivanon [14], i.e. $\text{Ric}^c(X) \cdot \varphi_0 = (X \lrcorner \sigma_T) \cdot \varphi_0$ and $\sigma_T \cdot \varphi_0 = -\frac{\text{Scal}^c}{4} \cdot \varphi_0$. Our most interesting result is related with ∇^c -parallel spinors. Such spinors have applications in theoretical physics, especially in type II string theory, where basic models are described in terms of a metric connection with skew-torsion and the corresponding parallel spinors represent the preserved supersymmetries (for more background we refer to [21, 14, 15]). In Section 4 we present the explicit action of the endomorphism $\text{Ric}^g(X)$ and more general $\text{Ric}^s(X)$ on $\text{Ker}(\nabla^c)$, where ∇^c is any metric connection with skew-torsion T such that $\nabla^c T = 0$ (this means, without assuming that ∇^c is the characteristic connection of some underlying special structure). In particular, we provide the following remarkable formula (see Theorem 4.7 and Corollaries 4.10, 4.13)

$$\begin{aligned} \text{Ric}^s(X) \cdot \varphi_0 &= \frac{(16s^2 - 1)}{4} \sum_{j=1}^n T(X, e_j) \cdot (e_j \lrcorner T) \cdot \varphi_0 + \frac{(16s^2 + 3)}{4} (X \lrcorner \sigma_T) \cdot \varphi_0 \\ &= \text{Ric}^c(X) \cdot \varphi_0 - \frac{(16s^2 - 1)}{4} S(X) \cdot \varphi_0, \end{aligned}$$

for any ∇^c -parallel spinor φ_0 and $X \in \Gamma(TM)$, where the endomorphism $S(X)$ is given by

$$S(X) := -X \lrcorner \sigma_T + \sum_{j=1}^n e_j \cdot (T(X, e_j) \lrcorner T).$$

Then, we specialise on some types of non-integrable geometric structures satisfying our assumptions, e.g. 5-dimensional Sasakian manifolds, 6-dimensional nearly Kähler manifolds and 7-dimensional nearly parallel G₂-manifolds. We illustrate our integrability conditions and describe the action of $\text{Ric}^s(X)$ on the corresponding ∇^c -parallel spinors (adapted to the particular special structure). For the Sasakian case, our result (see Theorem 4.16) nicely extends the integrability conditions given in [15, Thm. 7.3, 7.6], for any $s \in \mathbb{R}$. For

nearly Kähler structures and nearly parallel G_2 -structures we recover some of our conclusions in [9], however by a new method (see Corollary 4.14).

A final contribution of this note is related with the following first-order differential operator acting on spinors, $\not{D}^s(\varphi) := \sum_j (e_j \lrcorner T) \cdot \nabla_{e_j}^s \varphi$. This operator is included in the expression of $(D^s)^2$ and in the compact case can be viewed as the main obstruction to a universal estimate of the lowest eigenvalue of $(D^s)^2$, see [1, 14, 4]. In Section 5 we examine \not{D}^s and describe some special kinds of \not{D}^s -eigenspinors (see Proposition 5.2, 5.6). We also provide examples. Different classes of \not{D}^s -eigenspinors will be presented in a forthcoming work.

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2. PRELIMINARIES

With the aim to set up our conventions relevant to subsequent computations, we begin by recalling basic facts from spin geometry. Since all this material is well-known, we provide an exposition only of the most useful notions (without proofs) and for any further and detailed information, we refer to [11, 19, 14, 4, 2, 1].

2.1. Spin geometry with torsion. Consider an oriented connected Riemannian manifold (M^n, g) ($n \geq 3$) endowed with a spin structure, i.e. a $\text{Spin}(n)$ -principal bundle $\tilde{P}^g := \widetilde{\text{SO}}(M, g) \rightarrow M$ together with a 2-fold covering $\Lambda^g : \tilde{P}^g \rightarrow P^g$, such that $\Lambda^g(\tilde{p}g) = \Lambda^g(\tilde{p}) \text{Ad}(g)$ for any $\tilde{p} \in \tilde{P}^g$ and $g \in \text{Spin}_n$. Here, and for the following of this article we denote by $P^g := \text{SO}(M, g)$ the SO_n -principal bundle of positively oriented orthonormal frames of M . We also remind that for $n \geq 3$, the spin group Spin_n is the universal covering of SO_n and $\text{Ad} : \text{Spin}_n \rightarrow \text{SO}_n$ denotes the double covering map. Via the spin representation (which we agree to denote by κ_n), we associate to \tilde{P}^g a complex vector bundle $\Sigma^g M \rightarrow M$, the so-called spinor bundle $\Sigma^g M := \tilde{P}^g \times_{\kappa_n} \Delta_n = \tilde{P}^g \times_{\text{Spin}_n} \Delta_n$, where Δ_n is the spin module. Notice that the spinor bundle cannot be defined independently of a (semi)-Riemannian metric, in particular the definition of spinor fields, i.e. sections of the spinor bundle, depends in general on g , in contrast to tensors. For the following we set $\mathcal{F}^g := \Gamma(\Sigma^g M)$ and recall that $\Sigma^g M$ is endowed with a Spin_n -invariant Hermitian product $\langle \cdot, \cdot \rangle$, defined fiberwise as the natural Hermitian scalar product that admits Δ_n . Its real part $(\cdot, \cdot) := \text{Re} \langle \cdot, \cdot \rangle$ induces a positive definite inner product on $\Sigma^g M$. The Clifford multiplication is the bundle morphism $\mu : TM \otimes_{\mathbb{R}} \Sigma^g M \rightarrow \Sigma^g M$, defined by $\mu(X \otimes \varphi) := \kappa_n(X)(\varphi) = X \cdot \varphi$ and it naturally extends to differential forms $\mu : \Lambda(M) \otimes_{\mathbb{R}} \Sigma^g M \rightarrow \Sigma^g M$,

$$\omega \cdot \varphi := \sum_{1 \leq i_1 < \dots < i_p \leq n} \omega_{i_1 \dots i_p} e_{i_1} \cdot \dots \cdot e_{i_p} \cdot \varphi,$$

for any $\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} \omega_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p} \in \Lambda^p T^* M$. Given a vector field X , let us denote by X^\flat the dual 1-form, i.e. $X^\flat(u) = g(X, u)$. Then, any $X, Y \in T_x M$, $\omega \in \Lambda^p T_x^* M$ and $\varphi, \psi \in \Delta_n$ satisfy the following very useful properties (and similar for sections)

$$(2.1) \quad \left\{ \begin{array}{ll} -2g_x(X, Y)\varphi &= X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi \\ X \cdot \omega &= X^\flat \wedge \omega - X \lrcorner \omega \\ \omega \cdot X &= (-1)^p (X^\flat \wedge \omega + X \lrcorner \omega) \end{array} \right\} \left\| \begin{array}{ll} \langle X \cdot \varphi, \psi \rangle &= -\langle \varphi, X \cdot \psi \rangle \\ \langle \omega \cdot \varphi, \psi \rangle &= (-1)^{p(p+1)/2} \langle \varphi, \omega \cdot \psi \rangle \\ -2(X \lrcorner \omega) &= X \cdot \omega - (-1)^p \omega \cdot X. \end{array} \right.$$

From now on, let us assume that (M^n, g) is equipped with a non-trivial 3-form $T \in \Lambda^3 T^* M$. We consider the one-parameter family of metric connections $\{\nabla^s : s \in \mathbb{R}\}$ with skew-torsion $4sT$; this is defined by

$$g(\nabla_X^s Y, Z) = g(\nabla_X^g Y, Z) + 2sT(X, Y, Z),$$

for any $X, Y, Z \in \Gamma(TM)$. The family ∇^s lifts to a family of metric connections on $\Sigma^g M$, say $\nabla^s : \Gamma(\Sigma^g M) \rightarrow \Gamma(T^* M \otimes \Sigma^g M)$ (we keep the same notation), which explicitly reads by $\nabla_X^s \varphi = \nabla_X^g \varphi + s(X \lrcorner T) \cdot \varphi$. In terms of some local orthonormal frame $\{e_i\}$, it is $\nabla_X^s \varphi = \nabla_X^g \varphi + s \sum_{i < j} T(X, e_i, e_j) e_i \cdot e_j \cdot \varphi$ and the metric compatibility has the form $X \langle \varphi, \psi \rangle = \langle \nabla_X^s \varphi, \psi \rangle + \langle \varphi, \nabla_X^s \psi \rangle$. We also remind the Liebniz rule,

$$\nabla_X^s (Y \cdot \varphi) = (\nabla_X^s Y) \cdot \varphi + Y \cdot \nabla_X^s \varphi, \quad \nabla_X^s (\omega \cdot \varphi) = (\nabla_X^s \omega) \cdot \varphi + \omega \cdot \nabla_X^s \varphi,$$

for any $X, Y \in \Gamma(TM)$, $\omega \in \Lambda^p T^* M$ and $\varphi, \psi \in \mathcal{F}^g$.

The (spinorial) curvature operator $\mathcal{R}_{X,Y}^s := [\nabla_X^s, \nabla_Y^s] - \nabla_{[X,Y]}^s : \mathcal{F}^g \rightarrow \mathcal{F}^g$ associated to the covariant derivative ∇^s on the spinor bundle, satisfies the relation $\mathcal{R}_{X,Y}^s \varphi = \frac{1}{2} R^s(X \wedge Y) \cdot \varphi$, where R^s is the curvature operator on 2-forms, induced by ∇^s at the tangent bundle level. Locally, one has the relations

$$R^s(e_i \wedge e_j) := \sum_{k < l} R_{ijkl}^s e_k \wedge e_l, \quad \mathcal{R}_{X,Y}^s \varphi = \frac{1}{2} \sum_{i < j} g(R^s(X, Y) e_i, e_j) e_i \cdot e_j \cdot \varphi.$$

For $s = 0$, one obtains the (spinorial) Riemannian curvature operator $\mathcal{R}_{X,Y}^g$ and then, the first Bianchi identity associated to ∇^g yields the well-known relation

$$(2.2) \quad \frac{1}{2} \text{Ric}^g(X) \cdot \varphi = - \sum_{i=1}^n e_i \cdot \mathcal{R}_{X,e_i}^g \varphi,$$

where $\text{Ric}^g(X)$ is the Riemannian Ricci endomorphism, i.e. $\text{Ric}^g(X, Y) = g(\text{Ric}^g(X), Y)$, for any $X, Y \in \Gamma(TM)$. For our family ∇^s , and under the assumption $\nabla^c T = 0$, the associated Ricci tensor Ric^s remains symmetric (see also Remark 2.2 below) and the Ricci endomorphism $\text{Ric}^s(X)$ ($X \in \Gamma(TM)$) acts on spinors by the rule

$$\text{Ric}^s(X) \cdot \varphi := \sum_i g(\text{Ric}^s(X), e_i) e_i \cdot \varphi = \sum_i \text{Ric}^s(X, e_i) e_i \cdot \varphi.$$

In the same case, Becker-Bender proved that the first Bianchi identity associated to ∇^s (cf. [2, Thm. B.1]) induces an analogue of (2.2), namely:

Lemma 2.1. ([7, Lem. 1.13]) *Under the assumption $\nabla^c T = 0$, the following relation holds*

$$\frac{1}{2} \text{Ric}^s(X) \cdot \varphi = - \sum_i e_i \cdot \mathcal{R}_{X,e_i}^s \varphi + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi,$$

for any arbitrary vector field $X \in \Gamma(TM)$, spinor field $\varphi \in \mathcal{F}^g$ and $s \in \mathbb{R}$.

Remark 2.2. Let us denote by d^s and δ^s the differential and co-differential induced by ∇^s , i.e.

$$d^s \omega := \sum_j e_j \wedge \nabla_{e_j}^s \omega, \quad \delta^s \omega := - \sum_j e_j \lrcorner \nabla_{e_j}^s \omega,$$

respectively. For $s = 0$, we set $d := d^g \equiv d^0$ and $\delta := \delta^g \equiv \delta^0$. By [14, 4, 2] it is known that for some 3-form $T \in \Lambda^3 T^*M$, it is $\delta T = \delta^s T$ for any $s \in \mathbb{R}$. Notice also that the 4-form σ_T has a distinctive role in the theory of metric connections with skew-torsion. For example, by using [1, Lem. 2.4] one deduces that σ_T measures the difference of the differentials d^s and d , i.e. $d^s T = dT - 8s\sigma_T$. When the additional condition $\nabla^c T = 0$ holds, it is known that $dT = 2\sigma_T$ [21, 14] and $\nabla^c \sigma_T = 0 = \delta^s T$, for any $s \in \mathbb{R}$ [4, 2]. In the same case, it is also true that the curvature tensor R^s is symmetric, $R^s(X, Y, Z, W) = R^s(Z, W, X, Y)$ and the same holds for the associated Ricci tensor $\text{Ric}^s(X, Y) := \sum_i R^s(X, e_i, e_i, Y)$, i.e. $\text{Ric}^s(X, Y) = \text{Ric}^s(Y, X)$, for any $X, Y \in \Gamma(TM)$ and $s \in \mathbb{R}$, see [2, Thm. B.1].

We pass now on differential operators acting on spinors. The (*generalized*) *Dirac operator* is the first-order differential operator on \mathcal{F}^g , defined by

$$D^s := \mu \circ \nabla^s : \mathcal{F}^g \xrightarrow{\nabla^s} \Gamma(T^*M \otimes \Sigma^g M) \cong \Gamma(TM \otimes \Sigma^g M) \xrightarrow{\mu} \mathcal{F}^g.$$

The Spin_n -representation $\mathbb{R}^n \otimes \Delta_n$ splits as $\mathbb{R}^n \otimes \Delta_n = \ker(\mu) \oplus \Delta_n$ and this induces the decomposition $TM \otimes \Sigma^g M = \ker(\mu) \oplus \Sigma^g M$. We shall write $p : \mathbb{R}^n \otimes \Delta_n \rightarrow \ker(\mu)$ for the universal projection, which locally is defined by $X \otimes \varphi \mapsto X \otimes \varphi + \frac{1}{n} \sum_i e_i \otimes e_i \cdot X \cdot \varphi$. This naturally extends to sections and yields the (*generalized*) *Penrose, or twistor operator*,

$$\mathcal{P}^s = p \circ \nabla^s : \mathcal{F}^g \xrightarrow{\nabla^s} \Gamma(T^*M \otimes \Sigma^g M) \cong \Gamma(TM \otimes \Sigma^g M) \xrightarrow{p} \Gamma(\ker(\mu)).$$

Locally, the operators D^s and \mathcal{P}^s attain the expressions

$$D^s(\varphi) = \sum_{i=1}^n e_i \cdot \nabla_{e_i}^s \varphi, \quad \text{and} \quad \mathcal{P}^s(\varphi) := \sum_{i=1}^n e_i \otimes \left\{ \nabla_{e_i}^s \varphi + \frac{1}{n} e_i \cdot D^s(\varphi) \right\},$$

respectively. Finally we mention that the one-parameter family of generalized Dirac operators $\{D^s \equiv D^g + 3sT : s \in \mathbb{R}^*\}$ and the Riemannian Dirac operator $D^0 \equiv D^g$ are sharing several common properties.

For example, D^s is formally self-adjoint in $L^2(\Sigma^g M)$ for any $s \in \mathbb{R}$, since the torsion $T^s = 4sT$ is a 3-form [18]. Moreover, and similarly with the Riemannian case ($s = 0$) (cf. [11, 19]), one can show that:

Proposition 2.3. *For any $s \in \mathbb{R}$, $f \in C^\infty(M; \mathbb{R})$, $X \in \Gamma(TM)$, $\xi \in T^*M$, $\omega \in \Lambda^p T^*M$ and $\varphi \in \mathcal{F}^g$, the following hold:*

- (1) $D^s(f\varphi) = \text{grad}(f) \cdot \varphi + fD^s(\varphi)$.
- (2) *The principal symbol of D^s is given by $\sigma(D^s)(\xi)(\varphi) = \xi^\sharp \cdot \varphi$ and hence D^s is elliptic.*
- (3) *The operator $-(D^s)^2$ is strongly elliptic, i.e. $\langle \sigma(-(D^s)^2)(\xi)\varphi, \varphi \rangle = |\xi|^2 |\varphi|^2$.*
- (4) $D^s(X \cdot \varphi) = \sum_j e_j \cdot (\nabla_{e_j}^s X) \cdot \varphi - X \cdot D^s(\varphi) - 2\nabla_X^s \varphi$.
- (5) $D^s(w \cdot \varphi) = (-1)^p \omega \cdot D^s(\varphi) + (d^s \omega + \delta^s \omega) \cdot \varphi - 2 \sum_j (e_j \lrcorner \omega) \cdot \nabla_{e_j}^s \varphi$.

3. THE GENERALIZED $\frac{1}{2}$ -RICCI TYPE FORMULA AND BASIC APPLICATIONS

3.1. The generalized $\frac{1}{2}$ -Ricci type formula. In this section we shall introduce the generalized $\frac{1}{2}$ -Ricci type formula. For this is useful to fix, once and for all, a Riemannian spin manifold (M^n, g, T) ($n \geq 3$) endowed with a non-trivial 3-form $T \in \Lambda^3 T^*M$, such that $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{1}{2}T$. As we have already pointed out in Remark 2.2, the reason of our assumption $\nabla^c T = 0$ is the symmetry of the Ricci tensor Ric^s and Lemma 2.1.

Lemma 3.1. (The generalized $\frac{1}{2}$ -Ricci type formula, or $\frac{1}{2}$ -Ric^s-formula) *Assume that $\nabla^c T = 0$. Then, the Ricci endomorphism $\text{Ric}^s(X)$ satisfies*

$$\begin{aligned} \frac{1}{2} \text{Ric}^s(X) \cdot \varphi &= D^s(\nabla_X^s \varphi) - \nabla_X^s(D^s \varphi) - \sum_{j=1}^n e_j \cdot \left[\nabla_{\nabla_{e_j}^s X}^s \varphi + 4s \nabla_{T(X, e_j)}^s \varphi \right] \\ &\quad + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi, \end{aligned}$$

for any arbitrary vector field $X \in \Gamma(TM)$, spinor field $\varphi \in \mathcal{F}^g$ and $s \in \mathbb{R}$.

Proof. We use Lemma 2.1 and replace $\mathcal{R}_{X, e_j}^s \varphi$ by $\nabla_X^s \nabla_{e_j}^s \varphi - \nabla_{e_j}^s \nabla_X^s \varphi - \nabla_{[X, e_j]}^s \varphi$. This yields the following

$$\begin{aligned} \frac{1}{2} \text{Ric}^s(X) \cdot \varphi &= - \sum_j e_j \cdot \mathcal{R}_{X, e_j}^s \varphi + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi \\ &= - \sum_j e_j \cdot \left\{ \nabla_X^s \nabla_{e_j}^s \varphi - \nabla_{e_j}^s \nabla_X^s \varphi - \nabla_{[X, e_j]}^s \varphi \right\} + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi \\ &= \sum_j e_j \cdot (\nabla_{e_j}^s \nabla_X^s \varphi) - \sum_j e_j \cdot (\nabla_X^s \nabla_{e_j}^s \varphi) + \sum_j e_j \cdot \nabla_{[X, e_j]}^s \varphi + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi, \\ &= D^s(\nabla_X^s \varphi) - \sum_j \nabla_X^s(e_j \cdot \nabla_{e_j}^s \varphi) + \sum_j (\nabla_X^s e_j) \cdot (\nabla_{e_j}^s \varphi) + \sum_j e_j \cdot \nabla_{[X, e_j]}^s \varphi \\ &\quad + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi \\ &= D^s(\nabla_X^s \varphi) - \nabla_X^s(D^s \varphi) + \sum_j (\nabla_X^s e_j) \cdot (\nabla_{e_j}^s \varphi) + \sum_j e_j \cdot \nabla_{[X, e_j]}^s \varphi \\ &\quad + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi, \end{aligned}$$

where for the fourth equality we applied the Liebniz rule to replace $e_j \cdot (\nabla_X^s \nabla_{e_j}^s \varphi)$ by $\nabla_X^s(e_j \cdot \nabla_{e_j}^s \varphi) - (\nabla_X^s e_j) \cdot (\nabla_{e_j}^s \varphi)$. By the definition of T^s , we also have that $[X, e_j] = \nabla_X^s e_j - \nabla_{e_j}^s X - T^s(X, e_j) = \nabla_X^s e_j - \nabla_{e_j}^s X - 4sT(X, e_j)$ and consequently $\sum_j e_j \cdot \nabla_{[X, e_j]}^s \varphi = \sum_j e_j \cdot \left[\nabla_{\nabla_X^s e_j}^s \varphi - \nabla_{\nabla_{e_j}^s X}^s \varphi - 4s \nabla_{T(X, e_j)}^s \varphi \right]$. Hence, the formula given above reduces to the following one:

$$\begin{aligned} \frac{1}{2} \text{Ric}^s(X) \cdot \varphi &= D^s(\nabla_X^s \varphi) - \nabla_X^s(D^s \varphi) - \sum_j e_j \cdot \nabla_{\nabla_{e_j}^s X}^s \varphi + \sum_j \left[(\nabla_X^s e_j) \cdot (\nabla_{e_j}^s \varphi) + e_j \cdot \nabla_{\nabla_X^s e_j}^s \varphi \right] \\ &\quad - 4s \sum_j e_j \cdot \nabla_{T(X, e_j)}^s \varphi + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi. \end{aligned}$$

Now, it is easy to see that $\sum_j \left[(\nabla_X^s e_j) \cdot (\nabla_{e_j}^s \varphi) + e_j \cdot \nabla_{\nabla_X^s e_j}^s \varphi \right] = 0$, for any $s \in \mathbb{R}$, $X \in \Gamma(TM)$ and $\varphi \in \mathcal{F}^g$ (for the Riemannian case $s = 0$, see also [17, p. 133]). For example, this immediately follows after using (without loss of generality) a local ∇^s -parallel frame, i.e. a local orthonormal frame $\{e_j\}$ satisfying $(\nabla^s e_j)_x = 0$ at $x \in M$, for any $j = 1, \dots, n$. This finishes the proof. ■

Remark 3.2. Notice that the $\frac{1}{2}$ -Ric^s-formula (1.1) can be simplified a little further. Indeed, in terms of our ∇^s -parallel frame $\{e_i\}$ it is $[X, e_i] = -\nabla_{e_i}^s X - 4sT(X, e_i)$ and the third term in (1.1) reduces to

$$-\sum_{j=1}^n e_j \cdot \left[\nabla_{\nabla_{e_j}^s X}^s \varphi + 4s \nabla_{T(X, e_j)}^s \varphi \right] = \sum_{j=1}^n e_j \cdot \nabla_{[X, e_j]}^s \varphi.$$

Thus, an equivalent expression of the $\frac{1}{2}$ -Ric^s-identity is given by

$$(3.1) \quad \frac{1}{2} \text{Ric}^s(X) \cdot \varphi = D^s(\nabla_X^s \varphi) - \nabla_X^s(D^s \varphi) + \sum_{j=1}^n e_j \cdot \nabla_{[X, e_j]}^s \varphi + s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi.$$

In introduction we chose to present (1.1), instead of (3.1), since the reduction to (1.2) for $s = 0$ is direct. In fact, for the particular scopes of this work, relation (1.1) turns out to be more flexible than (3.1).

3.2. The new proof of the generalized SL-formula. Let us present now some basic applications of the new $\frac{1}{2}$ -Ric^s-formula, e.g. an alternative new proof of the following well-known and fundamental result (here we state the version for $\nabla^c T = 0$ only and refer to the cited articles for general skew-torsion).

Theorem 3.3. ([14, Thm. 3.1], [4, Thm. 6.1], [2, Thm. 2.1]) *Under the assumption $\nabla^c T = 0$, any spinor field $\varphi \in \mathcal{F}^g := \Gamma(\Sigma^g M)$ on (M^n, g, T) satisfies the relation*

$$(3.2) \quad (D^s)^2(\varphi) = \Delta^s(\varphi) + s(3 - 4s)dT \cdot \varphi - 4s\mathcal{D}^s(\varphi) + \frac{1}{4}\text{Scal}^s \cdot \varphi,$$

where \mathcal{D}^s is the first-order differential operator defined by $\mathcal{D}^s := \sum_i (e_i \lrcorner T) \cdot \nabla_{e_i}^s \varphi$ and $\Delta^s := (\nabla^s)^* \nabla^s := -\sum_i [\nabla_{e_i}^s \nabla_{e_i}^s + \nabla_{\nabla_{e_i}^s e_i}^s]$ denotes the spin Laplace operator associated to ∇^s .

First we recall that

Lemma 3.4. *Consider a p -form $\omega \in \Lambda^p T^*M$ and an orthonormal frame $\{e_j\}$. Then*

$$\sum_{j=1}^n e_j \cdot (e_j \lrcorner \omega) = p\omega, \quad \sum_{j=1}^n e_j \cdot (e_j \wedge \omega) = (p - n)\omega.$$

New proof of Theorem 3.3. Since the Ricci tensor Ric^s is symmetric, as in the Riemannian case, one can use the generalized $\frac{1}{2}$ -Ricci formula and apply a contraction with respect to a ∇^s -parallel local orthonormal frame $\{e_i\}$. This means

$$\sum_i e_i \cdot \text{Ric}^s(e_i) \cdot \varphi = \sum_{i,k} \text{Ric}^s(e_i, e_k) \cdot e_i \cdot e_k \cdot \varphi = -\sum_i \text{Ric}^s(e_i, e_i) \cdot \varphi = -\text{Scal}^s \cdot \varphi$$

and hence, for the left-hand side part of (1.1) we obtain $\frac{1}{2} \sum_i e_i \cdot \text{Ric}^s(e_i) \cdot \varphi = -\frac{1}{2} \text{Scal}^s \cdot \varphi$. Next we focus on the right-hand side and write all together:

$$\begin{aligned} -\frac{1}{2} \text{Scal}^s \cdot \varphi &= \sum_i e_i \cdot \left[D^s(\nabla_{e_i}^s \varphi) - \nabla_{e_i}^s(D^s \varphi) \right] - \sum_{i,j} e_i \cdot e_j \cdot \left[\nabla_{\nabla_{e_j}^s e_i}^s \varphi + 4s \nabla_{T(e_i, e_j)}^s \varphi \right] \\ &\quad + s(3 - 4s) \sum_i e_i \cdot (e_i \lrcorner \sigma_T) \cdot \varphi \\ &= \sum_i e_i \cdot D^s(\nabla_{e_i}^s \varphi) - \sum_i e_i \cdot \nabla_{e_i}^s(D^s \varphi) - \sum_{i,j} e_i \cdot e_j \cdot \left[\nabla_{\nabla_{e_j}^s e_i}^s \varphi + 4s \nabla_{T(e_i, e_j)}^s \varphi \right] \\ &\quad + 4s(3 - 4s) \sigma_T \cdot \varphi, \\ &= \sum_i e_i \cdot D^s(\nabla_{e_i}^s \varphi) - (D^s)^2(\varphi) - \sum_{i,j} e_i \cdot e_j \cdot \left[\nabla_{\nabla_{e_j}^s e_i}^s \varphi + 4s \nabla_{T(e_i, e_j)}^s \varphi \right] \\ &\quad + 4s(3 - 4s) \sigma_T \cdot \varphi, \end{aligned}$$

where we used the fact that $\sum_i e_i \cdot (e_i \lrcorner \sigma_T) = 4\sigma_T$ (see Lemma 3.4) and $-\sum_i e_i \cdot \nabla_{e_i}^s (D^s \varphi) = -(D^s)^2(\varphi)$. We proceed with the two sums appearing in the resulting formula. For the first one, we are based on the formula given in Proposition 2.3, (4); we replace X by e_i and φ by $\nabla_{e_i}^s \varphi$ and this yields

$$D^s(e_i \cdot \nabla_{e_i}^s \varphi) = \sum_i e_i \cdot (\nabla_{e_i}^s e_i) \cdot \nabla_{e_i}^s \varphi - e_i \cdot D^s(\nabla_{e_i}^s \varphi) - 2\nabla_{e_i}^s \nabla_{e_i}^s \varphi = -e_i \cdot D^s(\nabla_{e_i}^s \varphi) - 2\nabla_{e_i}^s \nabla_{e_i}^s \varphi.$$

Consequently,

$$\sum_i e_i \cdot D^s(\nabla_{e_i}^s \varphi) = -\sum_i D^s(e_i \cdot \nabla_{e_i}^s \varphi) - 2\sum_i \nabla_{e_i}^s \nabla_{e_i}^s \varphi = -(D^s)^2(\varphi) + 2\Delta^s(\varphi),$$

where in terms of the ∇^s -parallel local orthonormal frame $\{e_i\}$, the spin Laplace operator reads by $\Delta^s(\varphi) = (\nabla^s)^* \nabla^s \varphi = -\sum_i \nabla_{e_i}^s \nabla_{e_i}^s \varphi$ (observe that the relation $(\nabla^s e_i)_x = 0$, yields $(\nabla_{e_i}^s e_i)_x = 0$). Therefore, in order to complete the proof of Theorem 3.3, we just need to show that the second sum induces the operator \not{D}^s with the desired coefficient. Indeed, because $(\nabla_{e_j}^s e_i)_x = 0$, we obtain

$$-\sum_{i,j} e_i \cdot e_j \cdot \left[\nabla_{\nabla_{e_j}^s e_i}^s \varphi + 4s \nabla_{T(e_i, e_j)}^s \varphi \right] = -4s \sum_{i,j} e_i \cdot e_j \cdot \nabla_{T(e_i, e_j)}^s \varphi.$$

For a few, we forgot the factor $-4s$ and since $T(e_i, e_j) = \sum_k T_{ij}^k e_k = \sum_k T(e_i, e_j, e_k) e_k$, it follows that

$$\begin{aligned} \sum_{i,j} e_i \cdot e_j \cdot \nabla_{T(e_i, e_j)}^s \varphi &= \sum_{i,j} e_i \cdot e_j \cdot \nabla_{\sum_k T_{ij}^k e_k}^s \varphi \\ (3.3) \quad &= \sum_{i,j,k} T_{ij}^k e_i \cdot e_j \cdot \nabla_{e_k}^s \varphi = \sum_{i,j,k} T(e_i, e_j, e_k) e_i \cdot e_j \cdot \nabla_{e_k}^s \varphi, \end{aligned}$$

for some non-zero real numbers $T_{ij}^k := T(e_i, e_j, e_k) = g(T(e_i, e_j), e_k)$, with $T_{ij}^k = -T_{ji}^k = -T_{jk}^i$, etc. Recall now that $\sum_j T(X, e_j) \cdot e_j = -2(X \lrcorner T)$ for any $X \in \Gamma(TM)$ (see for example [2, p. 328]). Thus,

$$\begin{aligned} -2(e_k \lrcorner T) &= \sum_j T(e_k, e_j) \cdot e_j = \sum_{i,j} g(T(e_k, e_j), e_i) e_i \cdot e_j \\ (3.4) \quad &= \sum_{i,j} T(e_k, e_j, e_i) e_i \cdot e_j = -\sum_{i,j} T(e_i, e_j, e_k) e_i \cdot e_j, \end{aligned}$$

and a combination with (3.3) yields

$$(3.5) \quad \sum_{i,j} e_i \cdot e_j \cdot \nabla_{T(e_i, e_j)}^s \varphi = 2 \sum_k (e_k \lrcorner T) \cdot \nabla_{e_k}^s \varphi = 2\not{D}^s(\varphi).$$

Adding all our results together we obtain

$$-\frac{1}{2} \text{Scal}^s \cdot \varphi = -2(D^s)^2(\varphi) + 2\Delta^s(\varphi) - 8s\not{D}^s(\varphi) + 4s(3-4s)\sigma_T \cdot \varphi.$$

Since $\nabla^c T = 0$ it is $dT = 2\sigma_T$ and consequently the last identity is nothing than the generalized Schrödinger-Lichnerowicz formula under the condition $\nabla^c T = 0$. ■

3.3. The new proof of the twistorial $\frac{1}{2}$ -Ric^s-formula. Here we shall describe the restriction of the $\frac{1}{2}$ -Ric^s-formula to twistor spinors, providing a simpler proof of the twistorial $\frac{1}{2}$ -Ric^s-formula. This has been recently introduced by the author in [9, Lem. 2.2], with different however methods. Recall that a *twistor spinor with torsion* (TsT in short) is a (non-trivial) spinor field $\varphi \in \mathcal{F}^g$, solving the equation

$$\nabla_X^s \varphi + \frac{1}{n} X \cdot D^s(\varphi) = 0, \quad \text{for any } X \in \Gamma(TM),$$

for some parameter $s \neq 0$. Hence, a TsT is a spinor field belonging to the kernel of the generalized twistor operator \mathcal{P}^s , see [2, 9] for more details. Obviously, for $s = 0$ one obtains the usual notion of Riemannian twistor spinors, i.e. elements $\varphi \in \ker \mathcal{P}^g$, where $\mathcal{P}^g \equiv \mathcal{P}^0$ (cf. [11, 19]).

Theorem 3.5. (Twistorial $\frac{1}{2}$ -Ric^s-formula) *The action of the $\frac{1}{2}$ -Ric^s-formula on a non-trivial twistor spinor $\varphi \in \ker(\mathcal{P}^s)$ is given by*

$$\frac{1}{2} \text{Ric}^s(X) \cdot \varphi = \frac{1}{n} X \cdot (D^s)^2(\varphi) - \frac{n-2}{n} \nabla_X^s(D^s(\varphi)) + \frac{8s}{n} (X \lrcorner T) \cdot D^s(\varphi) + s(3-4s)(X \lrcorner \sigma_T) \cdot \varphi,$$

for any $X \in \Gamma(TM)$. In particular, for $s = 0$ and for a Riemannian twistor spinor, it induces the relation

$$\nabla_X^g(D^g(\varphi)) = \frac{n}{2(n-2)} \left[-\text{Ric}^g(X) \cdot \varphi + \frac{\text{Scal}^g}{2(n-1)} X \cdot \varphi \right] = \frac{n}{2} \text{Sch}^g(X) \cdot \varphi,$$

where $\text{Sch}^g(X) := \frac{1}{n-2} \left[-\text{Ric}^g(X) \cdot \varphi + \frac{\text{Scal}^g}{2(n-1)} X \right]$ is the Schouten endomorphism associated to ∇^g .

Proof. We apply the $\frac{1}{2}$ -Ric^s-formula (1.1) to a non-trivial twistor spinor $\varphi \in \ker(\mathcal{P}^s)$. Any vector field $X \in \Gamma(TM)$ satisfies $\nabla_X^s \varphi = -\frac{1}{n} X \cdot D^s(\varphi)$, hence for the first term in (1.1), Proposition 2.3, (4), yields that

$$\begin{aligned} D^s(\nabla_X^s \varphi) &= D^s\left(-\frac{1}{n} X \cdot D^s(\varphi)\right) = -\frac{1}{n} D^s(X \cdot D^s(\varphi)) \\ &= -\frac{1}{n} \left[\sum_j e_j \cdot (\nabla_{e_j}^s X) \cdot D^s(\varphi) - X \cdot D^s(D^s(\varphi)) - 2\nabla_X^s(D^s(\varphi)) \right] \\ &= -\frac{1}{n} \sum_j e_j \cdot (\nabla_{e_j}^s X) \cdot D^s(\varphi) + \frac{1}{n} X \cdot (D^s)^2(\varphi) + \frac{2}{n} \nabla_X^s(D^s(\varphi)). \end{aligned}$$

Therefore, for the first two terms of (1.1) we deduce that

$$(3.6) \quad D^s(\nabla_X^s \varphi) - \nabla_X^s(D^s(\varphi)) = \frac{1}{n} X \cdot (D^s)^2(\varphi) - \frac{n-2}{n} \nabla_X^s(D^s(\varphi)) - \frac{1}{n} \sum_j e_j \cdot (\nabla_{e_j}^s X) \cdot D^s(\varphi).$$

So, we have already constructed two desired terms with the right coefficients. Now, let us consider the sum $-\sum_{j=1}^n e_j \cdot \left[\nabla_{\nabla_{e_j}^s X}^s \varphi + 4s \nabla_{T(X, e_j)}^s \varphi \right]$ for some non-trivial $\varphi \in \ker(\mathcal{P}^s)$. Since $\nabla_{\nabla_{e_j}^s X}^s \varphi = -\frac{1}{n} (\nabla_{e_j}^s X) \cdot D^s(\varphi)$, it follows that

$$-\sum_{j=1}^n e_j \cdot \nabla_{\nabla_{e_j}^s X}^s \varphi = \frac{1}{n} \sum_{j=1}^n e_j \cdot (\nabla_{e_j}^s X) \cdot D^s(\varphi),$$

and this is canceled with the third term in the right-hand side of (3.6). Moreover, based on the identity $2(X \lrcorner T) = \sum e_j \cdot T(X, e_j)$ we obtain

$$-4s \sum_{j=1}^n e_j \cdot \nabla_{T(X, e_j)}^s \varphi = \frac{4s}{n} \sum_{j=1}^n e_j \cdot T(X, e_j) \cdot D^s(\varphi) = \frac{8s}{n} (X \lrcorner T) \cdot D^s(\varphi)$$

and our claim follows. For more details related to the case $s = 0$ we refer to [11, 19, 9]. ■

The twistorial $\frac{1}{2}$ -Ric^s-formula gives rise to

$$\frac{1}{2} \text{Scal}^s \cdot \varphi = -\frac{24s}{n} T \cdot D^s(\varphi) + \frac{2(n-1)}{n} (D^s)^2(\varphi) - 4s(3-4s) \sigma_T \cdot \varphi,$$

which it is not hard to see that is equivalent with the generalized SL-formula associated to the Dirac operator D^s , when this operator is restricted to $\ker(\mathcal{P}^s)$. The most basic consequences of the identity stated in Theorem 3.5, have been described in [9]. For example:

Proposition 3.6. ([9]) *Let (M^n, g, T) ($n \geq 3$) be a connected Riemannian spin manifold with $\nabla^c T = 0$. Then,*

- a) *The kernel of the twistor operator \mathcal{P}^s is finite dimensional, i.e. $\dim_{\mathbb{C}} \text{Ker}(\mathcal{P}^s) \leq 2^{\lfloor \frac{n}{2} \rfloor + 1}$.*
- b) *If φ and $D^s(\varphi)$ vanish at some point $p \in M$ and $\varphi \in \text{Ker}(\mathcal{P}^s)$, then $\varphi \equiv 0$.*

Obviously, Proposition 3.6 generalizes classical properties of Riemannian twistor spinors (cf. [11, 19]), to the whole family $\{\nabla^s : s \in \mathbb{R}\}$. Notice however that in the Riemannian case the space $\ker(\mathcal{P}^g)$ is in addition a conformal invariant of (M^n, g) . A similar result for connections with skew-torsion is known to hold only in dimension 4 (cf. [10]). For more details on TsT and interesting examples of special geometric structures admitting this kind of spinor fields, we refer to [7, 2, 9] (see also Section 4.2 below).

4. ∇^s -PARALLEL SPINORS AND ∇^c -PARALLEL SPINORS

4.1. Parallel spinors. In this section we present applications of the $\frac{1}{2}$ -Ric^s-formula, related with ∇^s -parallel spinors. Since Lemma 3.1 holds only under the assumption $\nabla^c T = 0$, it should be pointed out (even if this is not repeated throughout), that these results are meant for spin manifolds (M^n, g, T) with $\nabla^c T = 0$. Hence, fix a Riemannian spin manifold (M^n, g) endowed with a non-trivial 3-form $T \in \Lambda^3 T^* M$ such that $\nabla^c T = 0$, and consider the lift of the 1-parameter family $\{\nabla^s = \nabla^g + 2sT : s \in \mathbb{R}\}$ to the spinor bundle $\Sigma^g M$. By a ∇^s -parallel spinor we mean a non-trivial spinor field $\varphi_0 \in \mathcal{F}^g$ satisfying the equation $\nabla_X^s \varphi_0 = 0$, for some $s \in \mathbb{R}$ and any $X \in \Gamma(TM)$. This notion includes the following two well-known kinds of parallel spinors:

- $s = 0$; then we speak about ∇^g -parallel spinors and their existence yields the Ric^g-flatness of (M^n, g) , i.e. $\text{Ric}^g(X) \cdot \varphi_0 = 0$ for any $X \in \Gamma(TM)$, see for example [11, 19] (notice that an easy way to prove the Ricci flatness is via the $\frac{1}{2}$ -Ric^g-formula (1.2)).
- $s = 1/4$; then we speak about ∇^c -parallel spinors and is a simple consequence of [14, Col. 3.2] that when the condition $\nabla^c T = 0$ holds, then a solution of the relation $\nabla^c \varphi_0 = 0$ must satisfies the relations $\text{Ric}^c(X) \cdot \varphi_0 = (X \lrcorner \sigma_T) \cdot \varphi_0$, for any $X \in \Gamma(TM)$ and $\text{Scal}^c \cdot \varphi_0 = -4\sigma_T \cdot \varphi_0$.

The $\frac{1}{2}$ -Ric^s-formula immediately yields integrability conditions for any member of the family $\{\nabla^s : s \in \mathbb{R}\}$. Moreover, when a ∇^c -parallel spinor exists it allows us to describe the action of the endomorphism $\text{Ric}^s(X) : \mathcal{F}^g \rightarrow \mathcal{F}^g$ for any other s . We begin with the following:

Corollary 4.1. *Assume that $\nabla^c T = 0$ and let $\varphi_0 \in \mathcal{F}^g$ be a non-trivial spinor field which is parallel with respect to ∇^s , for some $s \in \mathbb{R}$. Then, for the same s and for any $X \in \Gamma(TM)$ the spinor φ_0 must satisfy the following:*

$$(4.1) \quad \text{Ric}^s(X) \cdot \varphi_0 = 2s(3 - 4s)(X \lrcorner \sigma_T) \cdot \varphi_0,$$

$$(4.2) \quad \text{Scal}^s \cdot \varphi_0 = -8s(3 - 4s)\sigma_T \cdot \varphi_0.$$

Remark 4.2. Notice that the connection $\nabla^{3/4}$ has torsion $3T$ and (4.1) shows that the existence of a $\nabla^{3/4}$ -parallel spinor φ_0 implies the Ric^{3/4}-flatness of (M^n, g, T) .

For the record, we also mention that

Corollary 4.3. *Assume that $\nabla^c T = 0$ and let φ_0 be a non-trivial ∇^s -parallel spinor for some $s \in \mathbb{R} \setminus \{0, 3/4\}$. Then, (M, g, T) is Ric^s-flat for the same parameter s , if and only if $(X \lrcorner \sigma_T) \cdot \varphi_0 = 0$ for any $X \in \Gamma(TM)$.*

Finally, since the 4-form σ_T vanishes in any dimension $n \leq 4$ ([14, 4]), we have that

Corollary 4.4. *A 3-dimensional or 4-dimensional Riemannian spin manifold (M^n, g, T) with $\nabla^c T = 0$, which admits a non-trivial ∇^s -parallel spinor $\varphi_0 \in \ker(\nabla^s)$ for some $s \in \mathbb{R}$, is Ric^s-flat for the same parameter s .*

Example 4.5. Consider the round 3-sphere (S^3, g_{can}) endowed with the volume form $T = \text{Vol}_{S^3}$. The characteristic connection $\nabla^c = \nabla^{\pm 1/4}$ (which is not unique because $S^3 \cong \text{Spin}_3 \cong \text{SU}_2$ is a Lie group), is induced by the Killing spinor equation. The real Killing spinors of S^3 trivialize its spinor bundle and they are ∇^c -parallel, see [4, p. 729]. Hence, Corollary 4.1 or Corollary 4.4 apply, and show that any such spinor $\{\varphi_j : 1 \leq j \leq 2^{\lfloor \frac{3}{2} \rfloor}\}$ must satisfy the equation $\text{Ric}^c(X) \cdot \varphi_j = 0$ for any $X \in \Gamma(TS^3)$, for another approach see for example [9, Prop. 5.1 and p. 133]. More general, any simply connected compact Lie group G with a bi-invariant metric g is flat with respect to the Cartan-Schouten connections $\nabla^{\pm 1/4}$ and there are $\nabla^{\pm 1/4}$ -parallel spinors which satisfy $\text{Ric}^{\pm 1/4}(X) \cdot \varphi = 0$ for any $X \in \mathfrak{g}$. Further Ric^c-flat structures carrying ∇^c -parallel spinors can be found in [4, 15, 13], for instance.

Remark 4.6. ([4]) In the compact case, Agricola and Friedrich [4, Thm. 7.1] proved that there are at most three parameters with ∇^s -parallel spinors. Indeed, assume that (M^n, g, T) is a compact Riemannian spin manifold endowed with a non-trivial 3-form $T \in \Lambda^3 T^* M$, such that $\nabla^c T = 0$. Then, any ∇^s -parallel spinor φ_0 of unit length satisfies

$$8s(3 - 4s) \int_M \langle \sigma_T \cdot \varphi_0, \varphi_0 \rangle v^g + \int_M \text{Scal}^s v^g = 0.$$

This follows immediately by integrating the condition (4.2). Based now on (5.2) one can show that if the mean value of $\langle \sigma_T \cdot \varphi_0, \varphi_0 \rangle$ does not vanish, then the parameter s equals to $s = 1/4$, i.e. φ_0 is necessarily parallel under the characteristic connection. If the mean value of $\langle \sigma_T \cdot \varphi_0, \varphi_0 \rangle$ vanishes, then the parameter s depends on Scal^g and $\|T\|^2$. We refer to [4] for further details and examples.

In the following we shall use the $\frac{1}{2}$ -Ric^s-formula to describe the spinorial action of the Ricci endomorphism $\text{Ric}^s(X)$ for any $s \in \mathbb{R}$, when there exists some ∇^c -parallel spinor φ_0 , without assuming however that ∇^c is the characteristic connection of some underlying special structure. An important fact for our approach is that the torsion T can be viewed as a $(\nabla^c$ -parallel) symmetric endomorphism on $\Sigma^g M$ in the sense that

$$(4.3) \quad \langle T \cdot \varphi, \psi \rangle = \langle \varphi, T \cdot \psi \rangle, \quad \forall \varphi, \psi \in \mathcal{F}^g.$$

Hence it is diagonalizable with real eigenvalues. Then, one may decompose the spinor bundle $\Sigma^g M$ into a direct sum of T -eigenbundles preserved by ∇^c , i.e. $\Sigma^g M = \bigoplus_{\gamma \in \text{Spec}(T)} \Sigma_\gamma^g M$ with $\nabla^c \Sigma_\gamma^g M \subset \Sigma_\gamma^g M$. This induces a splitting also to the space of sections, $\mathcal{F}^g = \bigoplus_{\gamma \in \text{Spec}(T)} \mathcal{F}^g(\gamma)$ with $\mathcal{F}^g(\gamma) := \Gamma(\Sigma_\gamma^g M)$. We finally remind that when the torsion T is ∇^c -parallel, then any non-trivial ∇^c -parallel spinor field has constant T -eigenvalues, i.e. the equations $\nabla^c T = 0$, $\nabla^c \varphi_0 = 0$ and $T \cdot \varphi_0 = \gamma \cdot \varphi_0$ for some $\gamma \in \text{Spec}(T)$ imply that $\gamma = \text{constant} \in \mathbb{R}$, see [5, Thm. 1.1].

Theorem 4.7. *Consider a Riemannian spin manifold (M^n, g, T) ($n \geq 3$) endowed with a non-trivial 3-form $T \in \Lambda^3 T^* M$, such that $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{1}{2}T$. Assume that φ_0 is a non-trivial ∇^c -parallel spinor field lying in $\mathcal{F}^g(\gamma)$, for some (constant) $\gamma \in \mathbb{R}$. Then, for any $s \in \mathbb{R}$ and $X \in \Gamma(TM)$ the following holds*

$$(4.4) \quad \begin{aligned} \text{Ric}^s(X) \cdot \varphi_0 &= -\frac{(16s^2 - 1)}{4} \sum_j e_j \cdot (T(X, e_j) \lrcorner T) \cdot \varphi_0 + \frac{(16s^2 + 3)}{4} (X \lrcorner \sigma_T) \cdot \varphi_0 \\ &= \frac{(16s^2 - 1)}{4} \sum_j T(X, e_j) \cdot (e_j \lrcorner T) \cdot \varphi_0 + \frac{(16s^2 + 3)}{4} (X \lrcorner \sigma_T) \cdot \varphi_0. \end{aligned}$$

Proof. The proof is rather long and relies on the $\frac{1}{2}$ -Ric^s-formula (1.1) and the ∇^c -parallelism of T . To begin with, notice that a ∇^c -parallel spinor φ_0 satisfies the following two equations (cf. [9])

$$(4.5) \quad \nabla_X^s \varphi_0 = \frac{4s - 1}{4} (X \lrcorner T) \cdot \varphi_0, \quad \text{and} \quad D^s(\varphi_0) = \frac{3(4s - 1)}{4} T \cdot \varphi_0,$$

for any $X \in \Gamma(TM)$. In particular φ_0 is a D^s -eigenspinor, $D^s(\varphi_0) = \frac{3(4s-1)\gamma}{4} \varphi_0$, for any $s \in \mathbb{R}$, with $D^c(\varphi_0) = 0$ of course. Let us apply the $\frac{1}{2}$ -Ric^s-formula to φ_0 . Due to (4.5) and Proposition 2.3, (5), for the first term of (1.1) we deduce that

$$\begin{aligned} D^s(\nabla_X^s \varphi_0) &= \frac{(4s - 1)}{4} D^s((X \lrcorner T) \cdot \varphi_0) \\ &= \frac{(4s - 1)}{4} \left[(X \lrcorner T) \cdot D^s(\varphi_0) + (d^s + \delta^s)(X \lrcorner T) \cdot \varphi_0 - 2 \sum_j (e_j \lrcorner X \lrcorner T) \cdot \nabla_{e_j}^s \varphi_0 \right] \\ &= 3 \left[\frac{(4s - 1)}{4} \right]^2 (X \lrcorner T) \cdot T \cdot \varphi_0 + \frac{(4s - 1)}{4} \left[d^s(X \lrcorner T) + \delta^s(X \lrcorner T) \right] \cdot \varphi_0 \\ &\quad - 2 \left[\frac{(4s - 1)}{4} \right]^2 \sum_j T(X, e_j) \cdot (e_j \lrcorner T) \cdot \varphi_0, \end{aligned}$$

where one identifies the 1-form $T(X, e_j)^\flat := g(T(X, e_j), -) = T(X, e_j, -) = e_j \lrcorner X \lrcorner T \in \Lambda^1 T^* M$ with its dual vector field $T(X, e_j) \in \Gamma(TM)$, via the metric tensor g . From (4.5) and since γ is a constant, we also obtain

$$\nabla_X^s (D^s(\varphi_0)) = 3 \left[\frac{(4s - 1)}{4} \right]^2 \gamma \cdot (X \lrcorner T) \cdot \varphi_0 = 3 \left[\frac{(4s - 1)}{4} \right]^2 (X \lrcorner T) \cdot T \cdot \varphi_0$$

and a combination with the stated expression for $D^s(\nabla^s \varphi_0)$, yields the difference

$$(4.6) \quad D^s(\nabla_X^s \varphi_0) - \nabla_X^s (D^s(\varphi_0)) = -\frac{(4s - 1)^2}{8} \sum_j T(X, e_j) \cdot (e_j \lrcorner T) \cdot \varphi_0 + \frac{(4s - 1)}{4} \left[d^s(X \lrcorner T) + \delta^s(X \lrcorner T) \right] \cdot \varphi_0.$$

We proceed with the action of the term $\mathcal{E}(\varphi_0) := -\sum_{j=1}^n e_j \cdot \left[\nabla_{\nabla_{e_j}^s X}^s \varphi_0 + 4s \nabla_{T(X, e_j)}^s \varphi_0 \right]$ on φ_0 . Because

$$\nabla_{\nabla_{e_j}^s X}^s \varphi_0 = \frac{4s-1}{4} ((\nabla_{e_j}^s X) \lrcorner T) \cdot \varphi_0, \quad \nabla_{T(X, e_j)}^s \varphi_0 = \frac{4s-1}{4} (T(X, e_j) \lrcorner T) \cdot \varphi_0,$$

and $\sum_j e_j \cdot (T(X, e_j) \lrcorner T) = -\sum_j T(X, e_j) \cdot (e_j \lrcorner T)$ (see [2, p. 325]), we conclude that

$$\begin{aligned} \mathcal{E}(\varphi_0) &= -\frac{(4s-1)}{4} \sum_j e_j \cdot ((\nabla_{e_j}^s X) \lrcorner T) \cdot \varphi_0 - s(4s-1) \sum_j e_j \cdot (T(X, e_j) \lrcorner T) \cdot \varphi_0 \\ (4.7) \quad &= -\frac{(4s-1)}{4} \sum_j e_j \cdot ((\nabla_{e_j}^s X) \lrcorner T) \cdot \varphi_0 + s(4s-1) \sum_j T(X, e_j) \cdot (e_j \lrcorner T) \cdot \varphi_0. \end{aligned}$$

Now, by the definition of d^s and since $\nabla_X^s(Y \lrcorner T) = (\nabla_X^s Y) \lrcorner T + Y \lrcorner (\nabla_X^s T)$ for any $X, Y \in \Gamma(TM)$ and $s \in \mathbb{R}$, it also follows that

$$d^s(X \lrcorner T) = \sum_j e_j \wedge ((\nabla_{e_j}^s X) \lrcorner T) = \sum_j e_j \wedge ((\nabla_{e_j}^s X) \lrcorner T) + \sum_j e_j \wedge (X \lrcorner (\nabla_{e_j}^s T)).$$

Having in mind the isomorphism $X \cdot \simeq X \wedge -X \lrcorner$, we combine the last relation with a part of (4.7), i.e.

$$\begin{aligned} \mathcal{A}(\varphi_0) &:= \frac{(4s-1)}{4} d^s(X \lrcorner T) \cdot \varphi_0 - \sum_{j=1}^n e_j \cdot \nabla_{\nabla_{e_j}^s X}^s \varphi_0 \\ (4.8) \quad &= \frac{(4s-1)}{4} \sum_j \left[e_j \lrcorner ((\nabla_{e_j}^s X) \lrcorner T) \right] \cdot \varphi_0 + \frac{(4s-1)}{4} \sum_j \left[e_j \wedge (X \lrcorner (\nabla_{e_j}^s T)) \right] \cdot \varphi_0. \end{aligned}$$

On the other hand, for any $X \in \Gamma(TM)$, $T \in \Lambda^3 T^*M$ and $s \in \mathbb{R}$ it holds that

$$\delta^s(X \lrcorner T) = -\sum_j e_j \lrcorner \nabla_{e_j}^s (X \lrcorner T) = -\sum_j e_j \lrcorner ((\nabla_{e_j}^s X) \lrcorner T) - \sum_j e_j \lrcorner (X \lrcorner (\nabla_{e_j}^s T)).$$

Therefore, adding appropriately with (4.8), the first sums cancel each other and we obtain

$$\begin{aligned} \mathcal{A}(\varphi_0) + \frac{(4s-1)}{4} \delta^s(X \lrcorner T) \cdot \varphi_0 &:= \frac{(4s-1)}{4} \left[d^s(X \lrcorner T) + \delta^s(X \lrcorner T) \right] \cdot \varphi_0 - \sum_{j=1}^n e_j \cdot \nabla_{\nabla_{e_j}^s X}^s \varphi_0 \\ &= \frac{(4s-1)}{4} \sum_j \left[e_j \wedge (X \lrcorner (\nabla_{e_j}^s T)) \right] \cdot \varphi_0 \\ &\quad - \frac{(4s-1)}{4} \sum_j \left[e_j \lrcorner (X \lrcorner (\nabla_{e_j}^s T)) \right] \cdot \varphi_0 \\ (4.9) \quad &= \frac{(4s-1)}{4} \sum_j e_j \cdot (X \lrcorner (\nabla_{e_j}^s T)) \cdot \varphi_0, \end{aligned}$$

where for the last equality we apply again the isomorphism $X \cdot \simeq X \wedge -X \lrcorner$. In this way and by combining the relations (4.6), (4.7), (4.8), (4.9) and (1.1), we conclude that

$$\begin{aligned} \frac{1}{2} \text{Ric}^s(X) \cdot \varphi_0 &= \frac{(4s-1)}{4} \sum_j e_j \cdot (X \lrcorner (\nabla_{e_j}^s T)) \cdot \varphi_0 + \frac{(16s^2-1)}{8} \sum_j T(X, e_j) \cdot (e_j \lrcorner T) \cdot \varphi_0 \\ (4.10) \quad &\quad + s(3-4s)(X \lrcorner \sigma_T) \cdot \varphi_0. \end{aligned}$$

The final step is based on the fact that under the condition $\nabla^c T = 0$, it holds that (see [2, Thm. B.1])

$$(\nabla_X^s T)(Y, Z, W) = \frac{4s-1}{2} \sigma_T(Y, Z, W, X) = -\frac{4s-1}{2} \sigma_T(X, Y, Z, W).$$

Thus, for any $X \in \Gamma(TM)$ the 3-form $(\nabla_X^s T)$ equals to

$$(\nabla_X^s T) = -\frac{4s-1}{2} (X \lrcorner \sigma_T)$$

and this has as a result the following:

$$\begin{aligned}
\frac{(4s-1)}{4} \sum_j e_j \cdot (X \lrcorner (\nabla_{e_j}^s T)) &= -\frac{(4s-1)^2}{8} \sum_j e_j \cdot (X \lrcorner (e_j \lrcorner \sigma_T)) \cdot \varphi_0 \\
&= \frac{(4s-1)^2}{8} \sum_j e_j \cdot (e_j \lrcorner (X \lrcorner \sigma_T)) \cdot \varphi_0 = \frac{(4s-1)^2}{8} \sum_j e_j \cdot (e_j \lrcorner \sigma_T^X) \cdot \varphi_0 \\
&= \frac{3(4s-1)^2}{8} \sigma_T^X \cdot \varphi_0 = \frac{3(4s-1)^2}{8} (X \lrcorner \sigma_T) \cdot \varphi_0.
\end{aligned}$$

Here, $\sigma_T^X := X \lrcorner \sigma_T$ is a 3-form, see also Lemma 3.4. In combination with (4.10), this observation completes the proof. ■

Remark 4.8. For the parameter $s = 1/4$, Theorem 4.7 reduces to $\text{Ric}^c(X) \cdot \varphi_0 = (X \lrcorner \sigma_T) \cdot \varphi_0$, for any $X \in \Gamma(TM)$, as it should be according to [14], or our Corollary 4.1.

For completeness, let us use (4.4) to verify the relation between the scalar curvatures Scal^s and Scal^g , namely $\text{Scal}^s = \text{Scal}^g - 24s^2 \|T\|^2$ (see [4]). Indeed, as in the proof of Theorem 3.3, we consider a local orthonormal frame $\{e_i\}$ and we write $\sum_i e_i \cdot \text{Ric}^s(e_i) \cdot \varphi_0 = -\text{Scal}^s \cdot \varphi$. On the other hand, it is $T(e_i, e_j) = \sum_k T_{ij}^k e_k = \sum_k T(e_i, e_j, e_k) e_k$ and by Theorem 4.7, for a non-trivial ∇^c -parallel spinor φ_0 , we see that

$$\begin{aligned}
\sum_i e_i \cdot \text{Ric}^s(e_i) \cdot \varphi_0 &= -\frac{(16s^2-1)}{4} \sum_{i,j} e_i \cdot e_j \cdot (T(e_i, e_j) \lrcorner T) \cdot \varphi_0 + \frac{(16s^2+3)}{4} \sum_i e_i \cdot (e_i \lrcorner \sigma_T) \cdot \varphi_0 \\
&\stackrel{(*)}{=} -\frac{(16s^2-1)}{4} \sum_{i,j,k} T_{ij}^k e_i \cdot e_j \cdot (e_k \lrcorner T) \cdot \varphi_0 + (16s^2+3) \sigma_T \cdot \varphi_0 \\
&\stackrel{(3.4)}{=} -\frac{(16s^2-1)}{2} \sum_k (e_k \lrcorner T) \cdot (e_k \lrcorner T) \cdot \varphi_0 + (16s^2+3) \sigma_T \cdot \varphi_0 \\
&\stackrel{(*)}{=} -\frac{(16s^2-1)}{2} [2\sigma_T - 3\|T\|^2] \cdot \varphi_0 + (16s^2+3) \sigma_T \cdot \varphi_0 \\
&= 4\sigma_T \cdot \varphi_0 + \frac{3(16s^2-1)}{2} \|T\|^2 \cdot \varphi_0 = -\text{Scal}^c \cdot \varphi_0 + \frac{3(16s^2-1)}{2} \|T\|^2 \cdot \varphi_0,
\end{aligned}$$

where for $(*)$ we used the fact $\sum_j (e_j \lrcorner T) \cdot (e_j \lrcorner T) = 2\sigma_T - 3\|T\|^2$ (see [2, p. 328]). We deduce that

$$\text{Scal}^s \cdot \varphi_0 = \text{Scal}^c \cdot \varphi_0 - \frac{3(16s^2-1)}{2} \|T\|^2 \cdot \varphi_0 = \text{Scal}^g \cdot \varphi_0 - 24s^2 \|T\|^2 \cdot \varphi_0.$$

and the assertion follows since φ_0 does not have zeros.

Remark 4.9. For these computations one could even proceed as follows: Based on (4.5), in $(*)$ we replace $(e_k \lrcorner T) \cdot \varphi_0$ by $\frac{4}{4s-1} \nabla_{e_k}^s \varphi_0$ for any $s \neq 1/4$. Then, we use (3.5) to obtain (a multiple of) the operator \mathcal{D}^s , appearing in Theorem 3.3. For the final step one needs a description of the \mathcal{D}^s -eigenvalues when this operator acts on ∇^c -parallel spinors, which we present in Section 5, see Proposition 5.2.

Corollary 4.10. Consider a triple (M^n, g, T) as in Theorem 4.7, admitting a non-trivial ∇^c -parallel spinor $\varphi_0 \in \mathcal{F}^g(\gamma)$ ($\gamma \in \mathbb{R}$). Then, the Riemannian Ricci endomorphism acts on φ_0 as

$$\begin{aligned}
\text{Ric}^g(X) \cdot \varphi_0 &= \frac{1}{4} \sum_j e_j \cdot (T(X, e_j) \lrcorner T) \cdot \varphi_0 + \frac{3}{4} (X \lrcorner \sigma_T) \cdot \varphi_0 \\
(4.11) \quad &= \frac{1}{8} \sum_j e_j \cdot (X \lrcorner T) \cdot (e_j \lrcorner T) \cdot \varphi_0 - \frac{3\gamma}{8} \cdot (X \lrcorner T) \cdot \varphi_0 + \frac{3}{4} (X \lrcorner \sigma_T) \cdot \varphi_0.
\end{aligned}$$

Proof. The first expression occurs by Theorem 4.7 for $s = 0$. The second one is based on the following lemma (observe that a similar reformulation as (4.11) applies also to Theorem 4.7 and hence the eigenvalue γ can appear in the corresponding expression, as well). ■

Lemma 4.11. *For any vector field $X \in \Gamma(TM)$, 3-form $T \in \Lambda^3 T^*M$ and orthonormal frame $\{e_j\}$, the following holds:*

$$\sum_{j=1}^n T(X, e_j) \cdot (e_j \lrcorner T) \equiv \sum_{j=1}^n (e_j \lrcorner X \lrcorner T) \cdot (e_j \lrcorner T) = -\frac{1}{2} \sum_{j=1}^n e_j \cdot (X \lrcorner T) \cdot (e_j \lrcorner T) + \frac{3}{2} (X \lrcorner T) \cdot T.$$

Proof. By (2.1) we see that $e_j \cdot \omega - (-1)^p \omega \cdot e_j = -2(e_j \lrcorner \omega)$ for any p -form $\omega \in \Lambda^p T^*M$. Since for any $X \in \Gamma(TM)$ the quantity $\omega := X \lrcorner T$ is a 2-form, it follows that $-\frac{1}{2} [e_j \cdot (X \lrcorner T) - (X \lrcorner T) \cdot e_j] = e_j \lrcorner X \lrcorner T$. Consequently, recalling that $\sum_j e_j \cdot (e_j \lrcorner T) = 3T$ (cf. [2, p. 328] or Lemma 3.4), one gets the result:

$$\begin{aligned} \sum_j (e_j \lrcorner X \lrcorner T) \cdot (e_j \lrcorner T) &= -\frac{1}{2} \sum_j [e_j \cdot (X \lrcorner T) - (X \lrcorner T) \cdot e_j] \cdot (e_j \lrcorner T) \\ &= -\frac{1}{2} \sum_j e_j \cdot (X \lrcorner T) \cdot (e_j \lrcorner T) + \frac{1}{2} \sum_j (X \lrcorner T) \cdot e_j \cdot (e_j \lrcorner T) \\ &= -\frac{1}{2} \sum_j e_j \cdot (X \lrcorner T) \cdot (e_j \lrcorner T) + \frac{3}{2} (X \lrcorner T) \cdot T. \quad \blacksquare \end{aligned}$$

Remark 4.12. As we explained in Remark 2.2, for ∇^c -parallel torsion T , the Ricci tensor satisfies the relation $\text{Ric}^s(X, Y) = \text{Ric}^g(X, Y) - 4s^2 S(X, Y)$, where S is a symmetric covariant 2-tensor defined by $S(X, Y) := \sum_i g(T(X, e_i), T(Y, e_i))$, see for example [9, p. 110]. Hence, the Ricci endomorphism $\text{Ric}^s(X)$ is given by $\text{Ric}^s(X) = \text{Ric}^g(X) - 4s^2 S(X)$, where $S(X)$ is the symmetric endomorphism associated to S , i.e. $g(S(X), Y) = S(X, Y)$, for any $X, Y \in \Gamma(TM)$. In particular, $\text{Ric}^g(X) = \text{Ric}^c(X) + \frac{1}{4} S(X)$ (cf. [14]). Therefore, a direct combination of Corollaries 4.10 and 4.1 for example, allows us to describe the explicit action of $S(X)$ ($X \in \Gamma(TM)$) on ∇^c -parallel spinors, for any metric connection $\nabla^c = \nabla^g + \frac{1}{2}T$ with $\nabla^c T = 0$.

Corollary 4.13. *Consider a triple (M^n, g, T) as in Theorem 4.7, admitting a non-trivial ∇^c -parallel spinor $\varphi_0 \in \mathcal{F}^g(\gamma)$ ($\gamma \in \mathbb{R}$). Then, for any $X \in \Gamma(TM)$, the action of the symmetric endomorphism $S(X)$ on φ_0 is given by*

$$\begin{aligned} S(X) \cdot \varphi_0 &= \sum_{j=1}^n e_j \cdot (T(X, e_j) \lrcorner T) \cdot \varphi_0 - (X \lrcorner \sigma_T) \cdot \varphi_0 \\ &= \frac{1}{2} \sum_{j=1}^n e_j \cdot (X \lrcorner T) \cdot (e_j \lrcorner T) \cdot \varphi_0 - \frac{3\gamma}{2} (X \lrcorner T) \cdot \varphi_0 - (X \lrcorner \sigma_T) \cdot \varphi_0. \end{aligned}$$

Theorem 4.7 and Corollaries 4.10 and 4.13 can be applied on any triple (M^n, g, T) endowed with a non-integrable G -structure $\mathcal{R} \subset P^g$ ($G \subsetneq \text{SO}_n$) and a ∇^c -parallel spinor φ_0 with respect to the adapted (unique) characteristic connection $\nabla^c = \nabla^g + \frac{1}{2}T$, under the assumption $\nabla^c T = 0$. Special structures fitting in this setting are plentiful, e.g. Sasakian manifolds in any odd dimension [14, 15], almost hermitian structures in even dimensions [6, 23], co-calibrated G_2 -structures in dimension 7 [14, 12], (non-parallel) Spin_7 -structures in dimension 8 [24, 20], to name some of them. In the following, we are going to illustrate our integrability conditions on nearly parallel G_2 -structures, nearly Kähler structures and Sasakian structures.

4.2. Real Killing spinors which are ∇^c -parallel. For the first two special structures mentioned above, the description can be globalized and this is because on these manifolds the existent ∇^c -parallel spinors coincide with the real Killing spinors, i.e. they satisfy the additional equation $\nabla_X^g \varphi_0 = \kappa X \cdot \varphi_0$ for any $X \in \Gamma(TM)$ and some $\kappa \in \mathbb{R}^*$ (with respect to the same metric g that holds $\nabla^c \varphi_0 = 0$). Friedrich and Ivanov [14, Thm. 5.6, 10.8] were the first who provided this identification and moreover proved that any such Einstein manifold is also ∇^c -Einstein. In [9, Prop. 5.1] we generalise these results by showing that the Ricci endomorphism $\text{Ric}^s(X)$ on such Einstein manifolds is a multiple of the identity operator for any $s \in \mathbb{R}$, i.e. $\text{Ric}^s = \frac{\text{Scal}^s}{n} \text{Id}$, and moreover that the existent ∇^c -parallel spinors are Killing spinor with torsion with respect to ∇^s (or twistor spinors with torsion) for any $s \in \mathbb{R} \setminus \{0, 1/4\}$ (for $s = 5/12$ and 6-dimensional nearly Kähler manifolds this result was known by [2, Thm. 6.1]). A direct and very simplified proof of the first conclusion arises now in terms of Theorem 4.7, as follows.

Let us denote by $\mathcal{K}^s(M, g)_\zeta := \{\varphi \in \mathcal{F}^g : \nabla_X^s \varphi = \zeta X \cdot \varphi \ \forall X \in \Gamma(TM)\}$ the set of all *Killing spinors with torsion* (KsT in short), with respect to the family $\nabla^s = \nabla^g + 2sT$ ($s \neq 0$), with Killing number $\zeta \neq 0$ (we refer to [2, 9] for a detailed exposition related to this kind of spinors). Similarly, we shall write $\mathcal{K}^0(M^n, g)_\kappa$ for the set of all real Killing spinors with Killing number $\kappa \neq 0$. Assume that (M^n, g, T) is a compact connected Riemannian spin manifold (M^n, g, T) , with $\nabla^c T = 0$ and positive scalar curvature given by $\text{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$, for some constant $0 \neq \gamma \in \text{Spec}(T)$. In [9, Thm. 3.7] we extended the identification mentioned above, namely

$$\text{Ker}(\nabla^c) \cong \bigoplus_{\gamma \in \text{Spec}(T)} \left[\Gamma(\Sigma_\gamma) \cap \mathcal{K}^0(M^n, g)_{\frac{3\gamma}{4n}} \right],$$

by proving that for any non-trivial ∇^c -parallel spinor φ_0 the following conditions are equivalent:

- (a) $\varphi_0 \in \mathcal{F}^g(\gamma) \cap \text{Ker}(\mathcal{P}^s) := \text{Ker}(\mathcal{P}^s|_{\Sigma_\gamma^g M})$ with respect to the family $\{\nabla^s : s \in \mathbb{R} \setminus \{1/4\}\}$,
- (b) $\varphi_0 \in \mathcal{K}^s(M, g)_\zeta$ with respect to the family $\{\nabla^s : s \in \mathbb{R} \setminus \{0, 1/4\}\}$ with $\zeta := 3(1 - 4s)\gamma/4n$,
- (c) $\varphi_0 \in \mathcal{K}^0(M, g)_\kappa$ with $\kappa := 3\gamma/4n$.

This correspondence allows us now to proceed with the following (see [9, Prop. 5.1] for another method):

Corollary 4.14. *Let (M^n, g, T) ($n \geq 3$) be a compact Riemannian spin manifold, endowed with the characteristic connection $\nabla^c = \nabla^g + \frac{1}{2}T$, such that $\nabla^c T = 0$. Assume that $0 \neq \varphi_0 \in \Sigma_\gamma^g M$ ($\mathbb{R} \ni \gamma \neq 0$) is a non-trivial ∇^c -parallel spinor, which satisfies one of the conditions (a), (b), or (c). Then, the $\frac{1}{2}$ -Ric^s-formula gives rise to the equation*

$$\text{Ric}^s(X) \cdot \varphi_0 = \frac{\text{Scal}^s}{n} X \cdot \varphi_0 = \frac{3\gamma^2(-3 + 3n - 144s^2 + 16ns^2)}{4n^2} X \cdot \varphi_0, \quad (\clubsuit)$$

for any $X \in \Gamma(TM)$, where $\text{Scal}^s := \frac{6\gamma^2}{n} \left[\frac{6(n-1)(1-4s)^2 + 96s(1-4s) + 16s(3-4s)(n-3)}{16} \right] = \frac{3\gamma^2(-3 + 3n - 144s^2 + 16ns^2)}{4n}$. Moreover, for $n \neq 9$ the symmetric endomorphism $S(X)$ acts on φ_0 as a multiple of the identity,

$$(4.12) \quad S(X) \cdot \varphi_0 = -\frac{3\gamma^2(n-9)}{n^2} X \cdot \varphi_0.$$

Proof. Assume that $0 \neq \varphi_0 \in \Sigma_\gamma^g M$ is a ∇^c -parallel spinor which satisfies any of the conditions (a), (b), or (c). Then, by [9, Prop. 3.2, Thm. 3.7] this is equivalent to say that φ_0 is a solution of the equation

$$(4.13) \quad (X \lrcorner T) \cdot \varphi_0 + \frac{3\gamma}{n} X \cdot \varphi_0 = 0,$$

for any $X \in \Gamma(TM)$. Moreover, the Ricci tensor Ric^c is computed algebraically (see for example the proof of [9, Prop. 5.1, (b)])

$$(4.14) \quad \text{Ric}^c(X) \cdot \varphi_0 = (X \lrcorner \sigma_T) \cdot \varphi_0 = \frac{3\gamma^2(n-3)}{n^2} X \cdot \varphi_0.$$

By (4.13) it follows that an arbitrary vector field X satisfies the equation $(T(X, e_j) \lrcorner T) \cdot \varphi_0 = -\frac{3\gamma}{n} T(X, e_j) \cdot \varphi_0$ and because $\sum_j e_j \cdot T(X, e_j) = 2(X \lrcorner T)$, an application of Theorem 4.7 gives rise to

$$\text{Ric}^s(X) \cdot \varphi_0 = -\frac{18\gamma^2(16s^2 - 1)}{4n^2} X \cdot \varphi_0 + \frac{3\gamma^2(n-3)(16s^2 + 3)}{4n^2} X \cdot \varphi_0,$$

which equals to the given relation. Finally, Corollary 4.13 yields the expression for the action of $S(X)$. ■

Therefore, as in [9], one concludes that a (complete) Riemannian manifold (M^n, g, T) ($n > 3$) satisfying Corollary 4.14 is a compact Einstein manifold with constant positive scalar curvature $\text{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$, a ∇^c -Einstein manifold with parallel torsion and constant positive scalar curvature $\text{Scal}^c = \frac{3(n-3)\gamma^2}{n}$ and moreover that satisfies the “harmony equation” (\clubsuit) for any other s . For $n = 3$, (M^3, g, T) is Ric^c-flat and hence isometric to the 3-sphere. Notice also that under our point of view, the study of 6-dimensional nearly Kähler manifolds and 7-dimensional nearly parallel G₂-manifolds reduces to be qualitatively the same. Therefore, below we illustrate our conclusions only for one of these two classes, e.g. nearly parallel G₂-manifolds and similarly is treated the former class (see also Example 5.5 and [14, 2, 9] for useful details).

Example 4.15. Consider a nearly parallel G_2 -manifold (M^7, g, ω) , i.e. a 7-dimensional oriented Riemannian manifold with a G_2 -structure $\omega \in \Gamma(\Lambda^3_+ T^*M)$ satisfying the differential equation $d\omega = -\tau_0 * \omega$, for some real constant $\tau_0 \neq 0$ (we refer to [16, 14, 12] for an introduction to G_2 -structures and also to G_2 -structures carrying a characteristic connection). In [14, Cor. 4.9] it was shown that a nearly parallel G_2 -manifold admits a unique characteristic connection ∇^c with *parallel* skew-torsion T , given by $T := \frac{1}{6}(d\omega, * \omega) \cdot \omega$, in particular $T = -\frac{\tau_0}{6}\omega$ and $\|T\|^2 = \frac{7}{36}\tau_0^2$. Moreover, there exists a unique spinor field φ_0 which is ∇^c -parallel (cf. [12, Prop. 3.2]) and satisfies the equation $T \cdot \varphi_0 = -\frac{7\tau_0}{6}\varphi_0 = -\sqrt{7}\|T\|\varphi_0$, i.e. $\gamma = -\sqrt{7}\|T\|$.¹ In fact, φ_0 is a real Killing spinor and hence $\text{Ric}^g(X) \cdot \varphi_0 = \frac{27}{14}\|T\|^2 X \cdot \varphi_0 = \frac{3\tau_0^2}{8} X \cdot \varphi_0$ ([16, 14]). More general, in [9, Exam. 5.3] we deduced the “harmony equation” (\heartsuit), i.e.

$$\text{Ric}^s(X) \cdot \varphi_0 = \frac{3(9 - 16s^2)}{14}\|T\|^2 X \cdot \varphi_0,$$

for any $s \in \mathbb{R}$ and $X \in \Gamma(TM)$. Let us provide a new proof of this result, via Theorem 4.7. As in the proof of Corollary 4.14, the key point is that φ_0 satisfies the equations (4.13) and (4.14) respectively, i.e.

$$(X \lrcorner T) \cdot \varphi_0 = \frac{\tau_0}{2} X \cdot \varphi_0 = \frac{3\|T\|}{\sqrt{7}} X \cdot \varphi_0, \quad \text{Ric}^c(X) \cdot \varphi_0 = (X \lrcorner \sigma T) \cdot \varphi_0 = \frac{12}{7}\|T\|^2 X \cdot \varphi_0,$$

for any $X \in \Gamma(TM)$. Both of them can be found [9] (see also [3, Lem. 2.3] and [14, p. 318]). The first represents the Killing equation, or equivalent the twistor equation [9, Prop. 3.2, Thm. 4.2], while the second one states that (M^7, g, ω) is a ∇^c -Einstein manifold. Hence, (4.4) yields that

$$\begin{aligned} \text{Ric}^s(X) \cdot \varphi_0 &= -\frac{3(16s^2 - 1)\|T\|}{4\sqrt{7}} \sum_{j=1}^7 e_j \cdot T(X, e_j) \cdot \varphi_0 + \frac{12(16s^2 + 3)\|T\|^2}{28} X \cdot \varphi_0 \\ &= -\frac{6(16s^2 - 1)\|T\|}{4\sqrt{7}} (X \lrcorner T) \cdot \varphi_0 + \frac{12(16s^2 + 3)\|T\|^2}{28} X \cdot \varphi_0 \\ &= -\frac{18(16s^2 - 1)\|T\|^2}{28} X \cdot \varphi_0 + \frac{12(16s^2 + 3)\|T\|^2}{28} X \cdot \varphi_0, \end{aligned}$$

which gives rises to the result. Finally, by (4.12) we compute $S(X) \cdot \varphi_0 = \frac{6\|T\|^2}{7} X \cdot \varphi_0$ (cf. [14]).

4.3. 5-dimensional Sasakian structures. Recall that a Sasakian structure on a Riemannian manifold (M^{2n+1}, g) consists of a Killing vector field ξ of unit length, the so-called Reeb vector field, such that the endomorphism $\phi : TM \rightarrow TM$ given by $\phi(X) = -\nabla_X^g \xi$, satisfies $(\nabla_X^g \phi)(Y) = g(X, Y)\xi - g(\xi, Y)X$ for any $X, Y \in \Gamma(TM)$. The dual 1-form η of ξ solves the equation $d\eta = 2F$, where $F(X, Y) := g(X, \phi(Y))$ is the fundamental 2-form, see for example [17, 8] for equivalent definitions and more details. Let us focus on 5-dimensional Sasakian manifolds $(M^5, g, \xi, \eta, \phi)$. We fix an orthonormal basis e_1, \dots, e_5 of $T_x M \simeq \mathbb{R}^5$ and use the abbreviation $e_{i_1 \dots i_p}$ for the p -form $e_{i_1} \wedge \dots \wedge e_{i_p}$. It is

$$\xi := e_5, \quad \phi := -(e_{12} + e_{34}), \quad F := e_{125} + e_{345},$$

and in terms of ϕ , our orthonormal frame reads by $\{e_1, e_2 := -\phi(e_1), e_3, e_4 = -\phi(e_3), e_5 = \xi\}$, with $\phi(\xi) = 0$. By [14, Prop. 7.1] it is known that there exists a unique metric connection ∇^c with *parallel* skew-torsion

$$T = \eta \wedge d\eta = 2\eta \wedge F = 2(e_{125} + e_{345}),$$

preserving the Sasakian structure, $\nabla^c g = \nabla^c \eta = \nabla^c \phi = 0$. The torsion form T acts on the 5-dimensional spin representation Δ_5 with eigenvalues $(-4, 0, 0, 4)$. Hence, the spinor bundle $\Sigma^g M$ splits into two 1-dimensional subbundles and one 2-dimensional subbundle, i.e. $\Sigma^g M = \Sigma_{-4}^g M \oplus \Sigma_0^g M \oplus \Sigma_4^g M$ with $\Sigma_{\pm 4}^g M := \{\varphi \in \Sigma^g M : T \cdot \varphi = \pm 4\varphi\}$ and $\Sigma_0^g M := \{\varphi \in \Sigma^g M : T \cdot \varphi = 0\}$, respectively.

In the direction of the Reeb vector field ξ the Riemannian Ricci endomorphism must occur with eigenvalue 4, i.e. $\text{Ric}^g(\xi) = 4\xi$ (cf. [17]). Let us explain how Corollary 4.10 fits with this result. Assume that there exists some ∇^c -parallel spinor φ_1 , which for instance belongs to $\Sigma_{-4}^g M$, i.e. $T \cdot \varphi_1 = -4\varphi_1$. Any vector field X satisfies $X \lrcorner F = -\phi(X)$, hence by (2.1) we get that (see also [17, Lem. 6.3])

$$(4.15) \quad X \cdot d\eta - d\eta \cdot X = -2(X \lrcorner d\eta) = -4(X \lrcorner F) = 4\phi(X), \quad \forall X \in \Gamma(TM).$$

¹Notice that here our 3-form ω is such that $\omega \cdot \varphi_0 = 7\varphi_0$, see [3, Lem. 2.3].

It is $\sigma_T = 4e_{1234}$, $\xi \lrcorner \sigma_T = 0$ and $\xi \lrcorner T = d\eta$. Thus, by applying for example (4.11) (for a description based on the first expression of Corollary 4.10, see the proof of Theorem 4.16 below), we obtain

$$\begin{aligned}
\text{Ric}^g(\xi) \cdot \varphi_1 &= \frac{1}{8} \sum_j e_j \cdot d\eta \cdot (e_j \lrcorner T) \cdot \varphi_1 + \frac{3}{2} d\eta \cdot \varphi_1 \\
&\stackrel{(4.15)}{=} \frac{1}{8} \sum_j d\eta \cdot e_j \cdot (e_j \lrcorner T) \cdot \varphi_1 + \frac{1}{2} \sum_j \phi(e_j) \cdot (e_j \lrcorner T) \cdot \varphi_1 + \frac{3}{2} d\eta \cdot \varphi_1 \\
&= \frac{3}{8} d\eta \cdot T \cdot \varphi_1 + \frac{1}{2} \sum_j \phi(e_j) \cdot (e_j \lrcorner T) \cdot \varphi_1 + \frac{3}{2} d\eta \cdot \varphi_1 = \frac{1}{2} \sum_j \phi(e_j) \cdot (e_j \lrcorner T) \cdot \varphi_1.
\end{aligned}$$

One also computes $e_2 \lrcorner T = -2e_{15}$, $e_1 \lrcorner T = 2e_{25}$, $e_4 \lrcorner T = -2e_{35}$, and $e_3 \lrcorner T = 2e_{45}$, which finally yields the desired assertion: $\text{Ric}^g(\xi) \cdot \varphi_1 = (-e_2 \cdot e_{25} - e_1 \cdot e_{15} - e_4 \cdot e_{45} - e_3 \cdot e_{35}) \cdot \varphi_1 = 4e_5 \cdot \varphi_1$. In fact, the relation $\text{Ric}^g(\xi) = 4\xi$ can be also obtained by using a ∇^c -parallel spinor in $\Sigma_4^g M$, or in $\Sigma_0^g M$. More general, in the simply-connected case one can use Corollary 4.10 to verify [14, Thm. 7.3, 7.6], i.e. the fact that the existence of a ∇^c -parallel spinor in $\Sigma_{\pm 4}^g M$ (resp. $\Sigma_0^g M$), requires that $(6, 6, 6, 6, 4)$ (resp. $(-2, -2, -2, -2, 4)$) are the eigenvalues of the Ricci endomorphism $\text{Ric}^g(X)$, and the converse. Next we are going to extend these known results, for any $s \in \mathbb{R}$, via Theorem 4.7. Notice that simply-connected Sasakian spin manifolds $(M^5, g, \xi, \eta, \phi)$ whose Ricci tensors Ric^g and Ric^c satisfy the prescribed curvature conditions, are for instance circle bundles over 4-dimensional Kähler-Einstein manifolds with positive scalar curvature and the 5-dimensional Heisenberg group (in fact, Sasakian structures $(M^5, g, \xi, \eta, \phi)$ with a ∇^c -parallel spinor in $\Sigma_0^g M$ are locally equivalent to the 5-dimensional Heisenberg group, see [14, 15]). These examples were described in [14, Examp. 7.4, 7.7] (see also [17, Examp. 6.1] and [15]) and for these Sasakian manifolds also the following more general theorem makes sense.

Theorem 4.16. *Consider a 5-dimensional simply-connected Sasakian spin manifold $(M^5, g, \xi, \eta, \phi)$ with its characteristic connection $\nabla^c = \nabla^g + \frac{1}{2}\eta \wedge d\eta = \nabla^g + \eta \wedge F$. Then,*

(1) *There exists a ∇^c -parallel spinor $\varphi_1 \in \Sigma_{-4}^g M$, or $\varphi_1 \in \Sigma_4^g M$, if and only if for any $s \in \mathbb{R}$ the eigenvalues of the Ricci tensor Ric^s are given by*

$$\{(6 - 32s^2), (6 - 32s^2), (6 - 32s^2), (6 - 32s^2), -4(16s^2 - 1)\}.$$

(2) *There exists a ∇^c -parallel spinor $\varphi_0 \in \Sigma_0^g M$, if and only if for any $s \in \mathbb{R}$ the eigenvalues of the Ricci tensor Ric^s are given by*

$$\{-(2 + 32s^2), -(2 + 32s^2), -(2 + 32s^2), -(2 + 32s^2), -4(16s^2 - 1)\}.$$

Proof. We begin again with the action of the endomorphism $\text{Ric}^s(\xi)$. As before, this is independent of which subbundle $\Sigma_\gamma^g M$ ($\gamma \in \{-4, 0, 4\}$) the ∇^c -parallel spinor is lying in. So, assume that ψ is a ∇^c -parallel spinor such that $\psi \in \Sigma_\gamma^g M$ for some (constant) $\gamma \in \mathbb{R}$. For the computation of the first term in Theorem 4.7, it is useful to remind that the Reeb vector field ξ is ∇^c -parallel, in particular ξ is a Killing vector field and consequently $\nabla_X^g \xi = \frac{1}{2}T(\xi, X) = -\phi(X) = \frac{1}{2}X \lrcorner d\eta$, see [15, 4]. Thus we compute

$$\nabla_{e_1}^g \xi = e_2, \quad \nabla_{e_2}^g \xi = -e_1, \quad \nabla_{e_3}^g \xi = e_4, \quad \nabla_{e_4}^g \xi = -e_3, \quad \nabla_\xi^g \xi = 0.$$

Let us also set $\mathcal{W} := -\sum_{j=1}^5 e_j \cdot (T(\xi, e_j) \lrcorner T) = \sum_{j=1}^5 T(\xi, e_j) \cdot (e_j \lrcorner T)$. Then we deduce that

$$\begin{aligned}
\mathcal{W} &= 2 \left[(\nabla_{e_1}^g \xi) \cdot (e_1 \lrcorner T) + (\nabla_{e_2}^g \xi) \cdot (e_2 \lrcorner T) + (\nabla_{e_3}^g \xi) \cdot (e_3 \lrcorner T) + (\nabla_{e_4}^g \xi) \cdot (e_4 \lrcorner T) \right] \\
&= 4 \left[e_2 \cdot e_{25} + e_1 \cdot e_{15} + e_4 \cdot e_{45} + e_3 \cdot e_{35} \right] = -16\xi.
\end{aligned}$$

One finishes with the first term, after a multiplication with the coefficient $(16s^2 - 1)/4$,

$$-\frac{16s^2 - 1}{4} \sum_{j=1}^5 e_j \cdot (T(\xi, e_j) \lrcorner T) \cdot \psi = \frac{16s^2 - 1}{4} \sum_{j=1}^5 T(\xi, e_j) \cdot (e_j \lrcorner T) \cdot \psi = -4(16s^2 - 1)\xi \cdot \psi.$$

Since $\xi \lrcorner \sigma_T = 0$, our claim follows, $\text{Ric}^s(\xi) \cdot \psi = -4(16s^2 - 1)\xi \cdot \psi$, for any $s \in \mathbb{R}$. Let us proceed now with the action of $\text{Ric}^s(X)$, for some $X \in \{e_1, \dots, e_4\}$. We analyse only the case $X = e_1$ and similarly are

treated the other vectors. At this point it is sufficient to assume that $\psi \in \Sigma_\gamma^g M$ ($\gamma \in \mathbb{R}$) is a ∇^c -parallel spinor (we use the fact that $\psi := \varphi_1 \in \Sigma_{\pm 4}^g M$, or $\psi := \varphi_0 \in \Sigma_0^g M$, only at the final step). We compute $T(e_1, e_1) = T(e_1, e_3) = T(e_1, e_4) = 0$ and $T(e_1, e_2) = 2e_5$, $T(e_1, \xi) = -T(\xi, e_1) = -2\nabla_{e_1}^g \xi = -2e_2$. Hence for the first term in Theorem 4.7 we deduce that

$$\begin{aligned} -\sum_j e_j \cdot (T(e_1, e_j) \lrcorner T) &= \sum_j T(e_1, e_j) \cdot (e_j \lrcorner T) = [T(e_1, e_2) \cdot (e_2 \lrcorner T) + T(e_1, \xi) \cdot (\xi \lrcorner T)] \\ &= -2[2e_5 \cdot e_{15} + e_2 \cdot d\eta] = -4[2e_1 + H], \end{aligned}$$

since inside Cl_5 we get $e_2 \cdot d\eta = 2(e_1 + H)$, where $H := e_{234} = \frac{1}{4}(e_1 \lrcorner \sigma_T)$. Multiplying with the coefficient $(16s^2 - 1)/4$, this gives rise to

$$-\frac{16s^2 - 1}{4} \sum_{j=1}^5 e_j \cdot (T(e_1, e_j) \lrcorner T) \cdot \psi = \frac{16s^2 - 1}{4} \sum_{j=1}^5 T(e_1, e_j) \cdot (e_j \lrcorner T) \cdot \psi = -(16s^2 - 1)[2e_1 + H] \cdot \psi.$$

Moreover, it is $\frac{(16s^2 + 3)}{4}(e_1 \lrcorner \sigma_T) \cdot \psi = (16s^2 + 3)H \cdot \psi$ and thus

$$\text{Ric}^s(e_1) \cdot \psi = 2(1 - 16s^2)e_1 \cdot \psi + 4H \cdot \psi.$$

The final step includes the action of the 3-form $H := e_{234} = \frac{1}{4}(e_1 \lrcorner \sigma_T)$ on $\Sigma_{\pm 4}^g M$ and $\Sigma_0^g M$, respectively, which of course is related with the endomorphism $\text{Ric}^c(e_1) = (e_1 \lrcorner \sigma_T)$ and is computed algebraically, see [14, pp. 324-325]:

$$H \cdot \psi = \begin{cases} e_1 \cdot \psi, & \text{if } \psi := \varphi_1 \in \Sigma_{\pm 4}^g M, \\ -e_1 \cdot \psi, & \text{if } \psi := \varphi_0 \in \Sigma_0^g M. \end{cases}$$

Consequently

$$(4.16) \quad \text{Ric}^s(e_1) \cdot \psi = \begin{cases} 2(1 - 16s^2)e_1 \cdot \psi + 4e_1 \cdot \psi = (6 - 32s^2)\psi, & \text{if } \psi := \varphi_1 \in \Sigma_{\pm 4}^g M, \\ 2(1 - 16s^2)e_1 \cdot \psi - 4e_1 \cdot \psi = -(2 + 32s^2)\psi, & \text{if } \psi := \varphi_0 \in \Sigma_0^g M, \end{cases}$$

for any $s \in \mathbb{R}$. For the converse, assume that $(M^5, g, \xi, \eta, \phi)$ is a 5-dimensional simply-connected Sasakian spin manifold whose Ricci tensor Ric^s ($s \in \mathbb{R}$) satisfies (4.16). Then, for $s = 0, 1/4$ we see that (4.16) induces the desired prescribed conditions, i.e. $\text{Ric}^g = \text{diag}(6, 6, 6, 6, 4)$, $\text{Ric}^c = \text{diag}(4, 4, 4, 4, 0)$ for $\varphi_1 \in \Sigma_{\pm 4}^g M$ and $\text{Ric}^g = \text{diag}(-2, -2, -2, -2, 4)$, $\text{Ric}^c = \text{diag}(-4, -4, -4, -4, 0)$ for $\varphi_0 \in \Sigma_0^g M$, respectively. Hence the assertion follows as in [14]. This finishes the proof. ■

Remark 4.17. The stated expressions of the Ricci endomorphism $\text{Ric}^s(X)$ can be also obtained by applying the general type $\text{Ric}^s = \text{Ric}^g - 4s^2 S = \text{Ric}^c - \frac{(16s^2 - 1)}{4} S$, where for the action of the symmetric endomorphism $S(X)$ on the related ∇^c -parallel spinor $\psi \in \Sigma_\gamma^g M$ one can apply Lemma 4.13. We refer also to [14] for $S(X)$.

5. ON THE DIFFERENTIAL OPERATOR $\mathcal{D}^s = \sum_i (e_i \lrcorner T) \cdot \nabla_{e_i}^s$

5.1. Special \mathcal{D}^s -eigenspinors. Next we examine some special eigenspinors of the differential operator

$$\mathcal{D}^s(\varphi) = \sum_i (e_i \lrcorner T) \cdot \nabla_{e_i}^s \varphi = \mathcal{D}^0(\varphi) + s \sum_i (e_i \lrcorner T) \cdot (e_i \lrcorner T) \cdot \varphi = \mathcal{D}^0(\varphi) + s\mathcal{T} \cdot \varphi,$$

appearing in Theorem 3.3, see [1, 4]. Here, \mathcal{D}^0 denotes the part corresponding to the Riemannian connection $\nabla^0 \equiv \nabla^g$ and $\mathcal{T} := \sum_j (e_j \lrcorner T) \cdot (e_j \lrcorner T) = 2\sigma_T - 3\|T\|^2$. Notice also by (3.5), that

$$\mathcal{D}^s(\varphi) = \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot \nabla_{T(e_i, e_j)}^s \varphi.$$

In fact, this formula holds also in the general case (although in the proof of Theorem 3.3 we use the assumption $\nabla^c T = 0$, this does not effect to the computations related to \mathcal{D}^s). By Proposition 2.3, (5) and Remark 2.2, one also has (see [14, 4, 2])

$$(5.1) \quad \mathcal{D}^s(\varphi) = -\frac{1}{2} [D^s(T \cdot \varphi) + T \cdot D^s(\varphi) - (dT + \delta T) \cdot \varphi + 8s\sigma_T \cdot \varphi].$$

Let us focus now on triples (M^n, g, T) with ∇^c -parallel skew-torsion, $\nabla^c T = 0$. In this case the operator \mathcal{D}^s has more equivalent expressions.

Lemma 5.1. ([4]) *Consider a Riemannian spin manifold (M^n, g, T) ($n \geq 3$) endowed with a non-trivial 3-form $T \in \Lambda^3 T^*M$, such that $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{1}{2}T$. Then, the operator \mathcal{D}^s is given by*

$$(5.2) \quad \mathcal{D}^s(\varphi) = -\frac{1}{2} \left[D^s(T \cdot \varphi) + T \cdot D^s(\varphi) - 2(1 - 4s)\sigma_T \cdot \varphi \right]$$

$$(5.3) \quad = -\frac{1}{2} \sum_j e_j \cdot T \cdot \nabla_{e_j}^s \varphi - \frac{1}{2} T \cdot D^s(\varphi),$$

where D^s is the (generalized) Dirac operator induced by ∇^s .

Proof. The first formula is an immediate consequence of (5.1). For the second description, we use (2.1), the definition of \mathcal{D}^s and relation (4.3). Then, for some arbitrary spinor fields φ, ψ we conclude that

$$\begin{aligned} \langle \mathcal{D}^s(\varphi), \psi \rangle &= -\frac{1}{2} \sum_j \langle e_j \cdot T \cdot \nabla_{e_j}^s \varphi, \psi \rangle - \frac{1}{2} \sum_j \langle T \cdot e_j \cdot \nabla_{e_j}^s \varphi, \psi \rangle \\ &= -\frac{1}{2} \sum_j \langle e_j \cdot T \cdot \nabla_{e_j}^s \varphi, \psi \rangle - \frac{1}{2} \sum_j \langle e_j \cdot \nabla_{e_j}^s \varphi, T \cdot \psi \rangle \\ &= -\frac{1}{2} \sum_j \langle e_j \cdot T \cdot \nabla_{e_j}^s \varphi, \psi \rangle - \frac{1}{2} \langle T \cdot D^s(\varphi), \psi \rangle, \end{aligned}$$

which gives rise to (5.3). ■

Therefore, when the torsion is ∇^c -parallel, it is $\sum_j e_j \cdot T \cdot \nabla_{e_j}^s \varphi = D^s(T \cdot \varphi) - 2(1 - 4s)\sigma_T \cdot \varphi = \text{grad}(\gamma) \cdot \varphi + \gamma D^s(\varphi) - 2(1 - 4s)\sigma_T \cdot \varphi$, where $\gamma \in \text{Spec}(T)$ denotes an eigenevalue of T , i.e. we assume (without loss of generality) that $\varphi \in \Sigma_\gamma^g M$ for some real function γ , not necessarily constant. Let us begin our investigation with ∇^c -parallel spinors, where γ is a real constant.

Proposition 5.2. *Consider a Riemannian spin manifold (M^n, g, T) ($n \geq 3$) with $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{1}{2}T$ is the metric connection with skew-torsion $0 \neq T \in \Lambda^3 T^*M$. Assume that $\varphi_0 \in \Sigma_\gamma^g M$ is a non-trivial ∇^c -parallel spinor and $\gamma \in \text{Spec}(T)$ is an eigenevalue of T . Then, φ_0 is an eigenspinor of the operator \mathcal{D}^s for any $s \in \mathbb{R}$,*

$$\mathcal{D}^s(\varphi_0) = -\frac{(4s-1)}{4} [T^2 + 2\|T\|^2] \cdot \varphi_0 = -\frac{(4s-1)}{4} [\gamma^2 + 2\|T\|^2] \varphi_0.$$

Proof. Based on (4.5) and the definition of \mathcal{D}^s , we see that any spinor field $\varphi \in \mathcal{F}^g$ satisfies

$$\mathcal{D}^s(\varphi) = \mathcal{D}^c(\varphi) + \frac{(4s-1)}{4} \mathcal{T} \cdot \varphi = \mathcal{D}^c(\varphi) + \frac{(4s-1)}{4} [2\sigma_T - 3\|T\|^2] \cdot \varphi,$$

where $\mathcal{D}^c := \sum_j (e_j \lrcorner T) \cdot \nabla_{e_j}^c$ is the operator associated to ∇^c . Thus, if $\nabla^c \varphi_0 = 0$, then $\mathcal{D}^c(\varphi_0) = 0$ and the claim immediately follows in combination with $\sigma_T \cdot \varphi_0 = \frac{1}{2}(\|T\|^2 - T^2) \cdot \varphi_0$ (cf. [4]). Of course, the same occurs by applying (5.3). Indeed, we rely again on (4.5) and compute that

$$\begin{aligned} \mathcal{D}^s(\varphi_0) &= -\frac{1}{2} \sum_j e_j \cdot T \cdot \nabla_{e_j}^s \varphi_0 - \frac{1}{2} T \cdot D^s(\varphi_0) \\ &= -\frac{4s-1}{8} \sum_j e_j \cdot T \cdot (e_j \lrcorner T) \cdot \varphi_0 - \frac{3(4s-1)}{8} T^2 \cdot \varphi_0. \end{aligned}$$

However, it is $e_j \cdot T = -T \cdot e_j - 2(e_j \lrcorner T)$, hence one can write

$$\begin{aligned} \mathcal{D}^s(\varphi_0) &= \frac{4s-1}{8} \sum_j T \cdot e_j \cdot (e_j \lrcorner T) \cdot \varphi_0 + \frac{2(4s-1)}{8} \sum_j (e_j \lrcorner T) \cdot (e_j \lrcorner T) \cdot \varphi_0 - \frac{3(4s-1)}{8} T^2 \cdot \varphi_0 \\ &= \frac{3(4s-1)}{8} T^2 \cdot \varphi_0 + \frac{(4s-1)}{4} (2\sigma_T - 3\|T\|^2) \cdot \varphi_0 - \frac{3(4s-1)}{8} T^2 \cdot \varphi_0 \\ &= \frac{(4s-1)}{4} (2\sigma_T - 3\|T\|^2) \cdot \varphi_0 = \frac{(4s-1)}{4} \mathcal{T} \cdot \varphi_0. \end{aligned}$$

Thus the assertion follows by using the relations $\mathcal{T} = -(T^2 + 2\|T\|^2)$ and $T^2 \cdot \varphi_0 = \gamma^2 \cdot \varphi_0$. ■

The action of the operator \mathcal{D}^s on Killing spinors and twistor spinors (with torsion or not), with respect to the family ∇^s , is known by [9]. In particular, for a non-trivial element $\varphi_0 \in \ker(\mathcal{P}^s)$ for some $s \in \mathbb{R}$ and independently of the assumption $\nabla^c T = 0$, it is not hard to show that

Proposition 5.3. ([9]) *Consider a Riemannian spin manifold (M^n, g, T) ($n \geq 3$) endowed with a non-trivial 3-form $T \in \Lambda^3 T^*M$ and the one-parameter family of metric connections $\nabla^s = \nabla^g + 2sT$. Then, any twistor spinor $\varphi_0 \in \ker(\mathcal{P}^s)$ (with torsion or not), with respect to ∇^s for some $s \in \mathbb{R}$, satisfies*

$$(5.4) \quad \mathcal{D}^s(\varphi_0) = -\frac{3}{n}T \cdot D^s(\varphi_0).$$

Moreover, if $\varphi_0 \in \mathcal{K}^s(M, g)_\zeta$ for some $s \in \mathbb{R} \setminus \{0, 1/4\}$ and $\zeta \neq 0$, then $\mathcal{D}^s(\varphi_0) = 3\zeta T \cdot \varphi_0$ and similarly, if $\varphi_0 \in \mathcal{K}^g(M, g)_\kappa$ for some $\kappa \neq 0$, then $\mathcal{D}^g(\varphi_0) = 3\kappa T \cdot \varphi_0$.

Corollary 5.4. *Whenever $\nabla^c T = 0$, a non-trivial KsT (resp. real Killing spinor) φ_0 induces a non-trivial eigenspinor of \mathcal{D}^s for the same s (resp. for $s = 0$) with eigenvalue $\beta := 3\gamma\zeta$, (resp. $\beta = 3\gamma\kappa$), where $\gamma \in \text{Spec}(T)$ is the corresponding T -eigenvalue.*

Example 5.5. Consider a 6-dimensional (strict) nearly Kähler manifold (M^6, g, J) , i.e. an almost Hermitian manifold endowed with a non-integrable almost complex structure J such that $(\nabla_X^g J)X = 0$. By [14, Thm. 10.1] it is known that M^6 admits a (unique) characteristic connection ∇^c with parallel skew-torsion, given by $T(X, Y) := (\nabla_X^g J)JY$. Moreover, there exist two ∇^c -parallel spinors φ^\pm such that $\mathcal{F}^g(\pm 2\|T\|)$, i.e. $T \cdot \varphi^\pm = \pm 2\|T\| \cdot \varphi^\pm$. Thus, by Proposition 5.2 we get

$$(5.5) \quad \mathcal{D}^s(\varphi^\pm) = -\frac{3(4s-1)\|T\|^2}{2}\varphi^\pm.$$

On the other hand, φ^\pm are real Killing spinors with $\kappa := \mp\|T\|/4$, TsT with torsion for any $s \neq 1/4$, i.e. $\varphi^\pm \in \text{Ker}(\mathcal{P}^s|_{\Sigma_{\pm 2\|T\|}^g M})$ and KsT for any $s \neq 0, 1/4$, with Killing number $\zeta := \mp\frac{(4s-1)}{4}\|T\|$, see [9, Thm. 4.1]. Therefore, (5.5) is deduced also by applying Corollary 5.4.

Under the condition $\nabla^c T = 0$, a kind of converse of Proposition 5.3 reads as follows:

Proposition 5.6. *Consider a triple (M^n, g, T) ($n \geq 3$) with $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{1}{2}T$ is the metric connection with skew-torsion $0 \neq T \in \Lambda^3 T^*M$. Assume that $\varphi_0 \in \Gamma(\Sigma_\gamma^g) \cap \text{Ker}(\mathcal{P}^s) := \text{Ker}(\mathcal{P}^s|_{\Sigma_\gamma^g M})$ is a non-trivial restricted twistor spinor (with torsion or not), for some $s_0 \in \mathbb{R}$ and some non-zero constant eigenvalue $0 \neq \gamma \in \text{Spec}(T)$. If φ_0 is a \mathcal{D}^{s_0} -eigenspinor, i.e. $\mathcal{D}^{s_0}(\varphi_0) = \beta\varphi_0$ for some constant eigenvalue β , then*

$$(5.6) \quad D^{s_0}(\varphi_0) = \frac{(n-6)\beta}{3\gamma}\varphi_0 + \frac{2(1-4s_0)}{\gamma}\sigma_T \cdot \varphi_0.$$

If $n = 6$ or $\beta = 0$, i.e. $\varphi_0 \in \ker(\mathcal{D}^{s_0})$, then $D^{s_0}(\varphi_0) = \frac{2(1-4s_0)}{\gamma}\sigma_T \cdot \varphi_0$.

Proof. Since $\mathcal{D}^{s_0}(\varphi_0) = \beta\varphi_0$ and $\varphi_0 \in \text{Ker}(\mathcal{P}^{s_0}|_{\Sigma_\gamma^g M})$, the type (5.4) reduces to $T \cdot D^{s_0}(\varphi_0) = -\frac{n\beta}{3}\varphi_0$ and since $\gamma \neq 0$ is a real constant such that $T \cdot \varphi_0 = \gamma \cdot \varphi_0$, our claim follows by relation (5.2). ■

Corollary 5.7. *If $\varphi_0 \in \text{Ker}(\mathcal{P}^c|_{\Sigma_\gamma^g M})$ is a non-trivial restricted twistor spinor with torsion with respect to ∇^c and φ_0 is \mathcal{D}^c -harmonic, i.e. $\beta = 0$ and hence $\mathcal{D}^c(\varphi_0) = 0$, then $D^c(\varphi_0) = 0$, in particular φ_0 is ∇^c -parallel, i.e. $\varphi_0 \in \ker(\nabla^c)$.*

Proof. This follows by Proposition 5.6 in combination with [9, Lem. 2.2], see also [9, p. 119] for details. ■.

We deduce that the relation $\varphi_0 \in \ker(\mathcal{D}^c) \cap \text{Ker}(\mathcal{P}^c|_{\Sigma_\gamma^g M})$ for some constant $\gamma \neq 0$, is a very strong condition which in fact implies the ∇^c -parallelism of φ_0 , similarly with the condition $\varphi_0 \in \ker(D^c) \cap \text{Ker}(\mathcal{P}^c|_{\Sigma_\gamma^g M})$. Hence, in general we avoid to consider this kind of TsT, as in [9].

Recall finally by Proposition 5.2 that for a ∇^c -parallel spinor field φ_0 the relation $\mathcal{D}^s(\varphi_0) = \beta\varphi_0$ is always verified with $\beta = \frac{(4s-1)}{4}[2\sigma_T - 3\|T\|^2]$. Adding now the extra condition $\varphi_0 \in \text{Ker}(\mathcal{P}^s|_{\Sigma_\gamma^g M})$ for some constant

$\gamma \neq 0$ and $s \neq 1/4$, then for $3 \leq n \leq 8$ we see that relation (5.6) gives rise to an alternative way to verify that φ_0 is actually a KsT with $\zeta = \frac{3(1-4s)}{4n}$ (see [9, Thm. 3.7]), i.e. $D^s(\varphi_0) = \frac{3(4s-1)\gamma}{4} \cdot \varphi_0$ as it should be according to (4.5). For such a proof one may use the formulas $\gamma^2 = \frac{2n}{9-n}\|T\|^2$ and $\sigma_T \cdot \varphi_0 = -\frac{3\gamma^2(n-3)}{4n}\varphi_0$, given in [9, Prop. 3.2]. For $n = 6$, relation (5.6) is simplified and we do not need the explicit form of β . This case of course applies on nearly Kähler manifolds. For nearly parallel G_2 -manifolds (M^7, g, ω) and for the unique ∇^c -parallel spinor $\varphi_0 \in \text{Ker}(\mathcal{P}^s|_{\Sigma_{-\sqrt{7}\|T\|}^g M})$ one computes $\beta = -\frac{9(4s-1)}{4}\|T\|^2$ via Proposition 5.2, hence (5.6) in combination with $\sigma_T \cdot \varphi_0 = -3\|T\|^2$ yield the result:

$$D^s(\varphi_0) = -\frac{21(4s-1)}{4\sqrt{7}}\|T\| \cdot \varphi_0 = \frac{3(4s-1)\gamma}{4} \cdot \varphi_0,$$

thus $\varphi_0 \in \mathcal{K}^s(M^7, g)_\zeta$ with $\zeta = -\frac{3(4s-1)}{4\sqrt{7}}\|T\|$ (cf. [9, Thm. 4.2]).

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